Partitioning Harary graphs into connected subgraphs containing prescribed vertices

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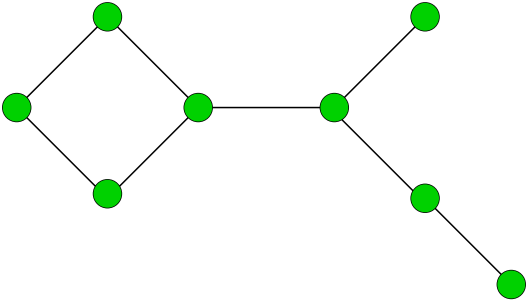
Part 1: Arbitrarily partitionable graphs

Our problem

Let us suppose we want to share the following network between several users in such a way that the subnetworks are connected and have the following sizes.

User 1: 1 User 2: 2 User 3: 2 User 4: 3



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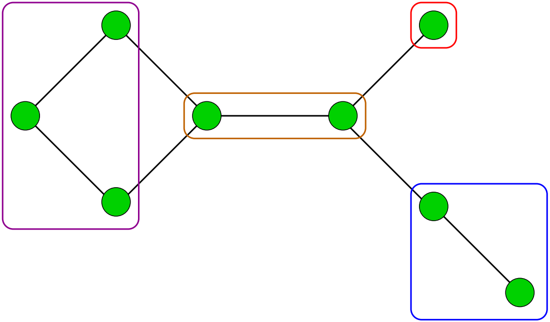

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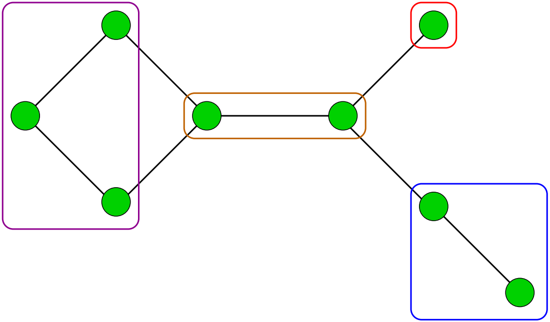

The more resource demands we can satisfy in a network, the more interesting it is.

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Let G be a connected graph on n vertices.

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Definition: *realizable sequence*, *realization*

A sequence $\tau = (\tau_1, ..., \tau_p)$ adding up to n is *realizable in G* if there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on τ_i vertices. The partition $(V_1, ..., V_p)$ is called a *realization of τ in G*.

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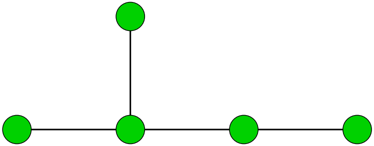
Definition: *AP graph*

If every sequence adding up to n is realizable in G, then G is said to be *arbitrarily partitionable*.

Networks with an AP graph topology are the most convenient regarding the previous problem when neither the number of users nor their needs are known beforehand.

An example of AP graph

Let us take a look at $Cat(2,3)$.



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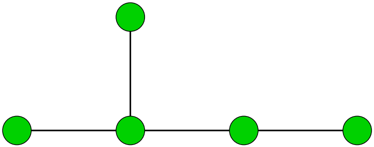

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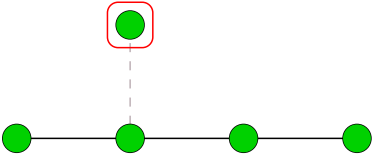

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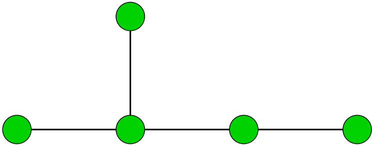

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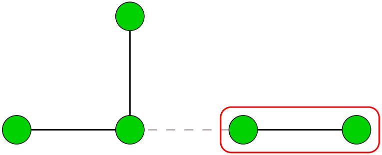

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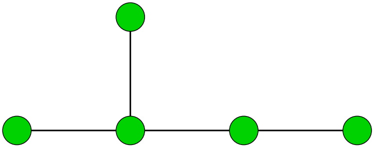

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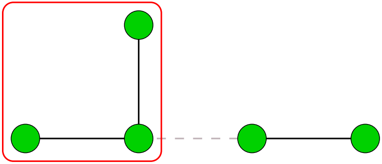

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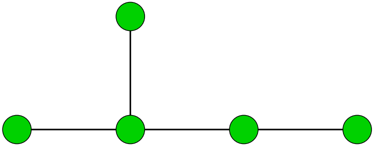

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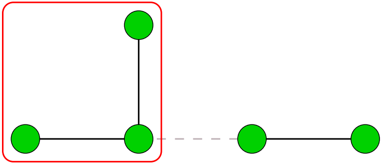
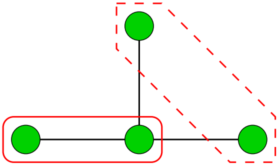


Since every non-trivial partition of 5 contains either a 1, a 2, or a 3, then $Cat(2,3)$ is AP.

Not all graphs are AP

The smallest non-AP graph is $Cat(2,2)$ (the claw) since it does not admit a realization of $(2,2)$.



About AP graphs

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... but deciding whether a graph is AP is difficult in general.

- This problem is NP-hard (Π_2^p-complete ?).
- Deciding whether a sequence is realizable in a graph is NPC [Rob98].
- There are $\Omega(e^{\sqrt{n}})$ partitions of n [FS09].

Stronger versions of this problem

Notice that our definition of AP graphs is not representative of the difficulties we can encounter while sharing a network.

- Our network is shared at once.
- The sharing is not performed until all of our resources are needed.
- The subnetworks resulting from the sharing are only connected.

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To deal with these deficiencies, some augmented versions of AP graphs were introduced.

- In the *online* version, the parts composing the partition of our graph are deduced one by one.
- In the *recursive* version, we want the subgraphs induced by a partition of our graph to be partitionable themselves.

Part 2: Partitioning graphs under prescriptions

Another partitioning constraint

Let us now suppose that our resources are not equivalent and that one of our users is allowed to request one specific resource to belong to his subnetwork.

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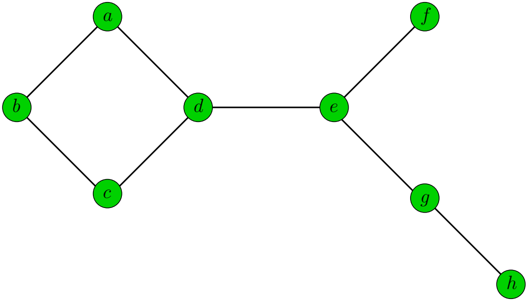

Well, let us try to satisfy this resource demand anyway...

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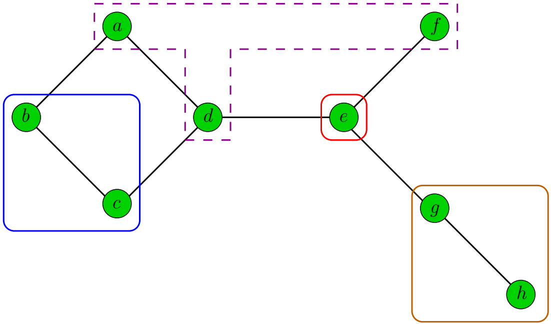

Sharing our network under these constraints is not possible here.

Formalization

Definition: *k-prescription, realization under prescription*

A k-tuple $(v_1, ..., v_k)$ of pairwise distinct vertices of G is called a *k-prescription of* G. If there exists a realization $(V_1, ..., V_p)$ in G of the sequence $\tau = (\tau_1, ..., \tau_p)$ with $p \geq k$ elements such that for every $i \in [1, k]$ we have $v_i \in V_i$, then τ is said to be *realizable in* G *under* P.

A sequence with several realizations in G may not be realizable in G following a given prescription. For example, there exists more than ten realizations of $(1, 2, 2, 3)$ in the previous graph but none of them admits $\{e\}$ as the part with size 1.

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Definition: *AP+k graph*

If every sequence adding up to n consisting of more than k elements is realizable in G under every k-prescription, then G is said to be *arbitrarily partitionable under k-prescriptions*.

Prescription and connectivity

This definition was inspired by the following well-known result.

Theorem (Lovász, 1977, and Györi, 1978, ind.) [Lov77, Gyo78]

A sequence $(\tau_1, \ldots, \tau_k)$ adding up to n is always realizable in a k-connected graph with order n under every k-prescription.

Caution: This result does not imply that every k-connected graph is AP+k!

Prescription and connectivity

However, a graph must be connected enough to be AP+k.

Observation

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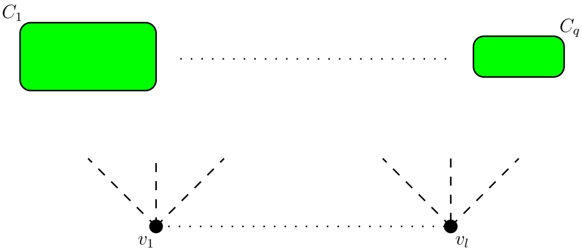

We consider that $l \leq k$.

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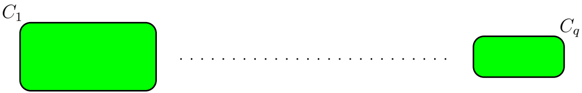

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Hence, we cannot realize $(1, ..., 1, |C_1|+1, (\sum_{i=2}^{q}|C_i|)-1)$ in this graph under $(v_1, ..., v_l)$. Finally, if $l < k$, then one has to prescribe some extra vertices to parts with size 1 until the prescription has size k.

Part 3: On the existence of AP+k graphs for arbitrary k

Partitioning powers of graphs

We prove the following two results.

Theorem 1 (Baudon, B., Przybyło, Woźniak, 2012)

The graph P_n^k is AP+$(k-1)$ for every $k \geq 1$ and $n \geq k$.

Theorem 2 (Baudon, B., Przybyło, Woźniak, 2012)

The graph C_n^k is AP+$(2k-1)$ for every $k \geq 1$ and $n \geq 2k$.

These results are sharp regarding the connectivity of the corresponding graphs.

Partitioning powers of paths

Lemma 1 (Baudon, B., Przybyło, Woźniak, 2012)

Let $P = (v_{i_1}, ..., v_{i_k})$ be a k-prescription of P_n^k with $k \geq 1$, $n \geq k$ and $i_1 < ... < i_k$. If i_k is the last vertex of P_n^k, then every partition $\tau = (\tau_1, ..., \tau_p)$ of n with $p \geq k$ elements is realizable in P_n^k under P.

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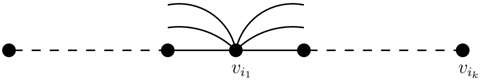
For arbitrary k, we use the following procedure to determine V_1 in such a way that the induction hypothesis can be used in $P_n^k - V_1$.



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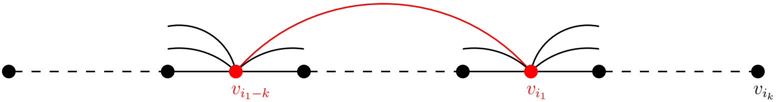
First, let $V_1 = \{v_{i_1}\}$. We then repeatedly "jump back at distance k" on the left of the last vertex added to V_1 as long as $|V_1| < \tau_1$ and the first vertex of P_n^k is not reached.



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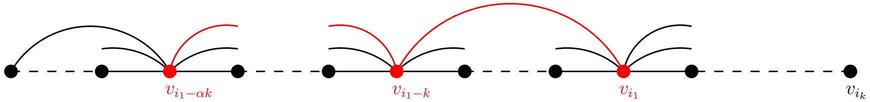

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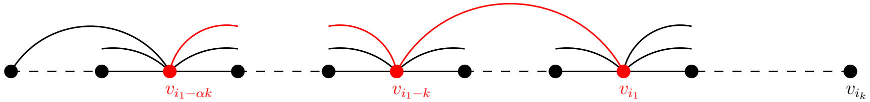

If, at one moment, we have $|V_1| = \tau_1$, then observe that we can use our induction hypothesis to deduce a realization of $(\tau_2, ..., \tau_p)$ in $P_n^k - V_1$ under $(v_{i_2}, ..., v_{i_k})$. It follows that $(V_1, ..., V_p)$ is a whole realization of τ in P_n^k under P.

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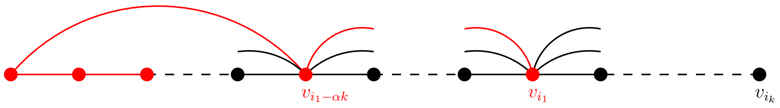
Otherwise, we add to V_1 every remaining vertex of $\{v_1, ..., v_{i_1 - 1}\} - V_1$ from left to right as long as $|V_1| < \tau_1$ holds.



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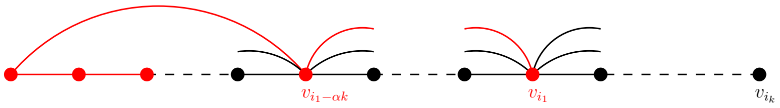

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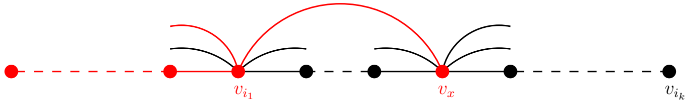
If V_1 still does not have size τ_1, then let $r \in \{0, ..., k-1\} - (\bigcup_{j=2}^{k-1} i_j \mod k)$. We then add v_x to V_1, where v_x is a neighbour of v_{i_1} such that $x > i_1$ and $x \equiv r \mod k$.



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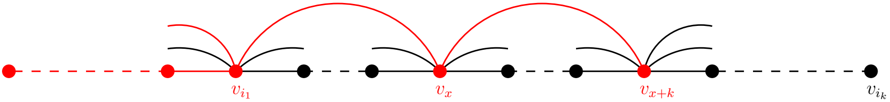

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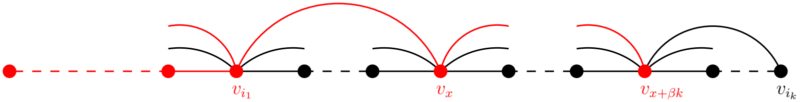

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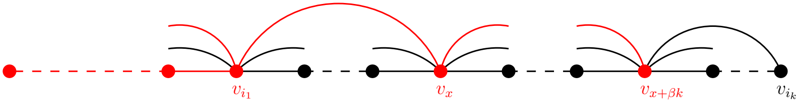

If V_1 has size τ_1 at one moment, then the previous statements can be used once again to deduce the realization.

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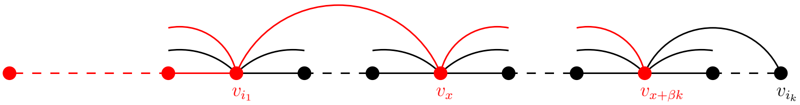

After this procedure, every vertex of $V - V_1$ has a neighbour in V_1 and $P_n^k - V_1$ is the $(k-1)^{th}$ power of a path. Thus, according to our induction hypothesis, there exists a realization $(V_2, \ldots, V_p, V_1')$ of $(\tau_2, \ldots, \tau_p, \tau_1 - |V_1|)$ in $P_n^k - V_1$ under $(v_{i_2}, \ldots, v_{i_k})$. Finally, $(V_1 \cup V_1', V_2, \ldots, V_p)$ is a realization of τ in P_n^k under P. ■

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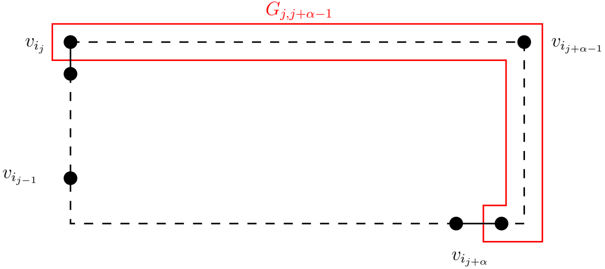
Given α consecutive prescribed vertices $v_{i_j}, \ldots, v_{i_{j+\alpha-1}}$, the *garden of* $v_{i_j}, \ldots, v_{i_{j+\alpha-1}}$ *in* C_n^k is the subset $G_{j,j+\alpha-1} = \{v_{i_j}, \ldots, v_{i_{j+\alpha}-1}\}$ of consecutive vertices of C_n^k.



In particular, observe that $C_n^k[G_{x,y}]$ is the k^{th} power of a path.

Partitioning powers of cycles

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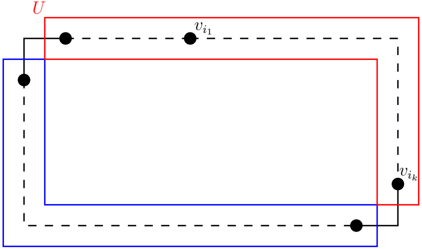

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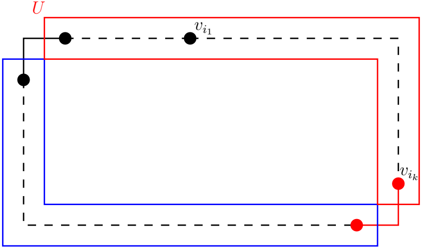

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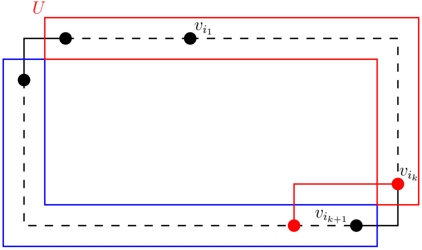

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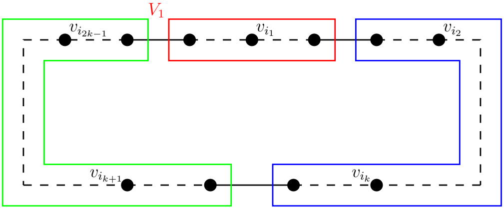

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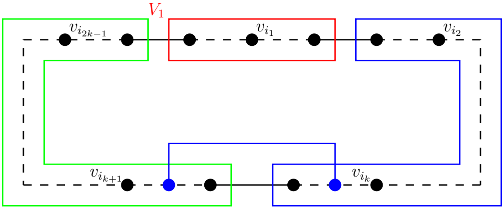

Adding edges to AP+k graphs

Observe that the property of being AP+k is closed under edge-addition too. Starting from the k^{th} power of paths and cycles, we get the following.

Corollary of Theorems 1 and 2

The k^{th} power of a traceable or Hamiltonian graph is AP+$(k-1)$ or $AP + (2k-1)$, respectively.

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Hence, we now focus on AP+k graphs having the least number of edges.

Part 4: On optimal AP+k graphs

A lower bound on the size of an AP+k graph

Recall that an AP+k graph must be $(k+1)$-connected.
Hence, we deduce the following.

Observation

If G is an AP+k graph on n vertices, then $\|G\| \geq \lceil \frac{n(k+1)}{2} \rceil$.

An AP+k whose size meets this lower bound is called an *optimal AP+k graph*.

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We here only focus on the **existence** of optimal AP+k graphs on n vertices for every $k \geq 1$ and $n \geq k$.

Harary graphs

Harary provided a construction which yields a k-connected graph with order n whose size is $\lceil \frac{kn}{2} \rceil$ for arbitrary k and n.

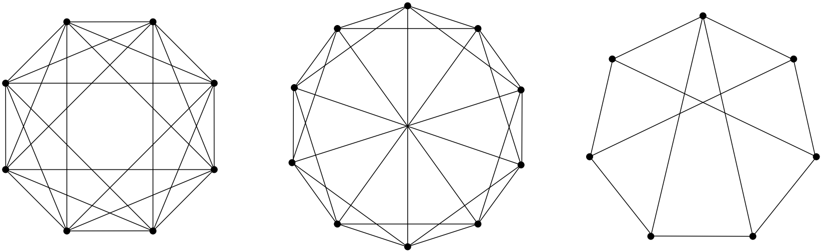
Definition: *Harary graph*

Let $k \geq 1$ and $n \geq k$ be any two integers. The k-connected Harary graph on n vertices, denoted by $H_{k,n}$, has vertex set $\{v_0, \ldots, v_{n-1}\}$ and the following edges:

- if $k = 2r$ is even, then two vertices v_i and v_j are linked if $i - r \leq j \leq i + r$;
- if $k = 2r + 1$ is odd and n is even, then $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in [0, \frac{n}{2} - 1]$;
- if $k = 2r + 1$ and n are odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$;

where the subscripts are taken modulo n.

Some examples of Harary graphs



The Harary graphs $H_{6,8}$, $H_{5,10}$, and $H_{3,7}$

Partitioning Harary graphs with even connectivity

Harary graphs are Hamiltonian, and thus are AP. Then, how many prescriptions can be made before partitioning $H_{k,n}$?

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Observe that, for even k, the graph $H_{k,n}$ is isomorphic to $C_n^{k/2}$.

Corollary of Theorem 2

The Harary graph $H_{k,n}$ is AP+$(k-1)$ for every even k.

Partitioning Harary graphs with odd connectivity

Observe that $H_{2k+1,n}$ with $2k+1$ odd is spanned by C_n^k and thus is AP+$(2k-1)$ according to Theorem 2. Although this number of prescriptions is good regarding the connectivity of $H_{2k+1,n}$ we would like to allow one more prescription while partitioning it.

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We now sketch the proof of the following result.

Theorem 3 (Baudon, B., Sopena, 2012)

The Harary graph $H_{2k+1,n}$ is AP+$2k$ for every $k \neq 1$.

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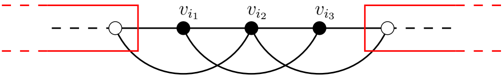
In $H_{2k+1,n}$, the prescribed blocks with size at least k alter the original structure of the graph.



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Theorem 3 (Baudon, B., Sopena, 2012)

The Harary graph $H_{2k+1,n}$ is AP+$2k$ for every $k \neq 1$.

We distinguish three main cases depending on the number and the sizes of the prescribed blocks.

1. There is no prescribed block with size at least k.
2. There is exactly one prescribed block with size at least k.
3. There are two prescribed blocks with size k.

In the first two cases, a realization can be deduced in the underlying C_n^k of $H_{2k+1,n}$, while we need to use the *diagonal edges* of $H_{2k+1,n}$ to handle the third case.

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The Harary graph $H_{2k+1,n}$ is AP+$2k$ for every $k \neq 1$.

If P is a $2k$-prescription of C_n^k with at most one prescribed block in C_n^k with size at least k, then every sequence can be realized in C_n^k under P. This statement can be proved using the following two lemmas.

Lemma 2 (Baudon, B., Sopena, 2012)

Let $P = (v_{i_1}, ..., v_{i_{k+1}})$ be a $(k+1)$-prescription of P_n^k with $k \geq 1$, $n \geq k$ and $i_1 < ... < i_{k+1}$. If i_1 and i_{k+1} are the first and last vertices of P_n^k, respectively, then every partition $\tau = (\tau_1, ..., \tau_p)$ of n with $p \geq (k+1)$ elements is realizable in P_n^k under P.

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Partitioning Harary graphs with odd connectivity

Theorem 3 (Baudon, B., Sopena, 2012)

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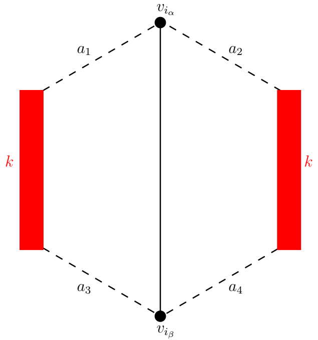
Now suppose that there are two prescribed blocks with size k. There necessarily exists a diagonal edge incident with two non-prescribed vertices.



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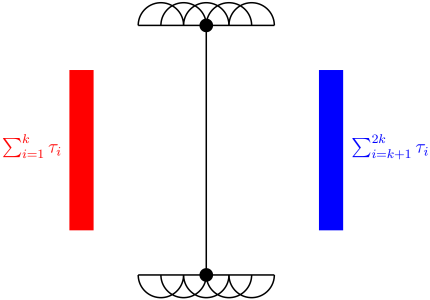

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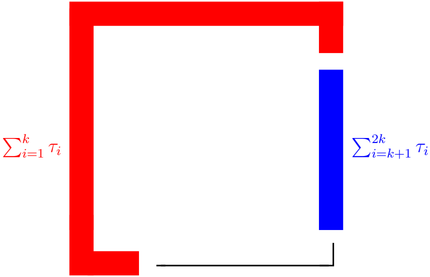
- If $\sum_{i=1}^{2k} \tau_i \geq a_1 + a_2 + 2k + 1$, then we use Györi-Lovász Theorem again.



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- If $\sum_{i=1}^{2k} \tau_i \geq a_1 + a_2 + 2k + 1$, then we use Györi-Lovász Theorem again.
- Otherwise, we have $\sum_{i=1}^{2k} \tau_i \geq a_3 + a_4 + 2k + 1$ and the same strategy is applicable.

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What about optimal AP+2 graphs?

The proof of Theorem 3 uses the fact that some subgraphs of $H_{2k+1,n}$ are traceable whenever $k > 1$. Clearly, this argument does not hold when $k = 1$. Therefore, our proof does not hold to prove that $H_{3,n}$ is AP+2 for arbitrary n.

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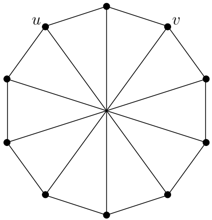
The Harary graph $H_{3,n}$ is not AP+2 when $n \equiv 2 \mod 4$.



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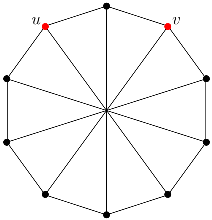

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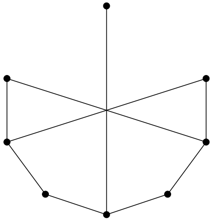

This subgraph has no perfect matching. Thus, $H_{3,10}$ does not admit a realization of $(1, 1, 2, 2, 2, 2)$ under (u, v).

On the existence of optimal AP+2 graphs

Recall that P_n can be arbitrarily partitioned under (v_1, v_n) as long as v_1 and v_n are the endvertices of P_n. Thanks to a spanning graph argument, we get the following.

Corollary of Lemma 2

A Hamiltonian-connected graph is AP+2.

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A Hamiltonian-connected graph is AP+2.

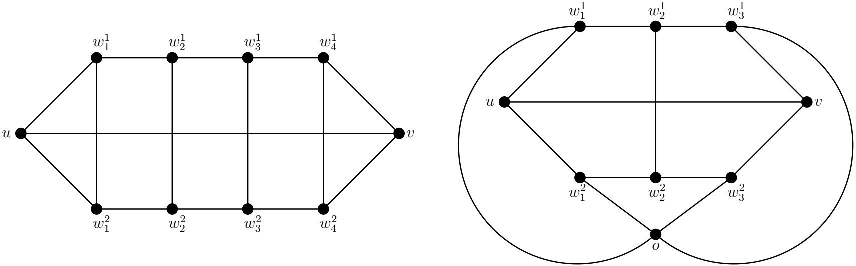
Using this sufficient condition, one can prove that the following graphs are AP+2 for every n.

Definition: *Pr_n graphs*

Let $n \geq 4$. The graph Pr_n is constructed as follows:

- If n is even, Pr_n is obtained from the cycle C_n, whose vertices are successively denoted by $u, w_1^1, ..., w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, ..., w_1^2$, by adding it the edge uv and all edges $w_i^1 w_i^2$, for every $i \in [1, \frac{n-2}{2}]$.
- If n is odd, Pr_n is obtained by first removing the edges $w_1^1 w_1^2$ and $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$ from Pr_{n-1}, and then adding it a new vertex o linked to w_1^1, w_1^2, $w_{\frac{n-3}{2}}^1$, and $w_{\frac{n-3}{2}}^2$.

Examples of Pr_n graphs



The graphs Pr_{10} and Pr_9

Pr_n graphs are AP+2

Proposition (Baudon, B., Sopena, 2012)

The graph Pr_n is Hamiltonian-connected for every $n \geq 6$.

s	t	P
u	v	$uP^{\nearrow}_{1,q}(Pr_n)v$
u	w_i^1	$uP^{\nearrow}_{1,i-1}(Pr_n)w_i^2P^{2,\rightarrow}_{i+1,q}(Pr_n)vP^{1,\leftarrow}_{q,i}(Pr_n)$ if $i-1$ is even $uP^{\searrow}_{1,i-1}(Pr_n)w_i^2P^{2,\rightarrow}_{i+1,q}(Pr_n)vP^{1,\leftarrow}_{q,i}(Pr_n)$ otherwise
w_i^1	w_j^1	$P^{1,\rightarrow}_{i,j-1}(Pr_n)P^{2,\leftarrow}_{j-1,i}(Pr_n)P^{\nwarrow}_{i-1,1}(Pr_n)uvP^{\nwarrow}_{q,j}(Pr_n)$ if $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(Pr_n)P^{2,\leftarrow}_{j-1,i}(Pr_n)P^{\nwarrow}_{i-1,1}(Pr_n)uvP^{\swarrow}_{q,j}(Pr_n)$ otherwise
w_i^1	w_j^2	$P^{1,\rightarrow}_{i,j-1}(Pr_n)P^{2,\leftarrow}_{j-1,i}(Pr_n)P^{\nwarrow}_{i-1,1}(Pr_n)uvP^{\swarrow}_{q,j}(Pr_n)$ if $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(Pr_n)P^{2,\leftarrow}_{j-1,i}(Pr_n)P^{\nwarrow}_{i-1,1}(Pr_n)uvP^{\nwarrow}_{q,j}(Pr_n)$ otherwise

Existence of a Hamiltonian path in Pr_n whose endvertices are s and t, for n even and where $q = \frac{n-2}{2}$

Such Hamiltonian paths also exist when n is odd...

Concluding result

Corollary (Baudon, B., Sopena, 2012)

There exist optimal AP+k graphs on n vertices for every $k \geq 1$ and $n \geq k$.

References I

O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak.
Partitioning powers of traceable or hamiltonian graphs.
Preprint, 2012.

O. Baudon, J. Bensmail, and É. Sopena.
Partitioning harary graphs into connected subgraphs containing prescribed vertices.
Preprint, 2012.

P. Flajolet and R. Sedgewick.
Analytic combinatorics.
Cambridge University Press, 2009.

E. Gyori.
On division of graphs to connected subgraphs.
In *Combinatorics*, pages 485–494, Colloq. Math. Soc. János Bolyai 18, 1978.

References II

L. Lovász.
A homology theory for spanning trees of a graph.
Acta Math. Acad. Sci. Hung., 30(3-4):241–251, 1977.

M. Robson.
Private communication, 1998.