A Decompositional Approach to the 1-2-3 Conjecture

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*Université Côte d'Azur, France

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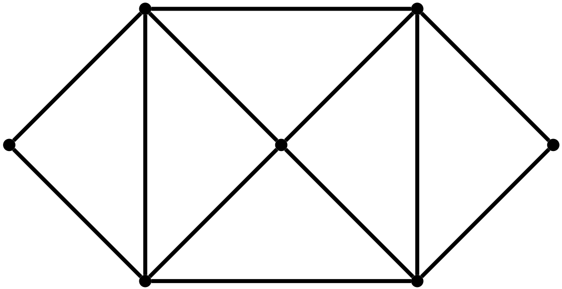
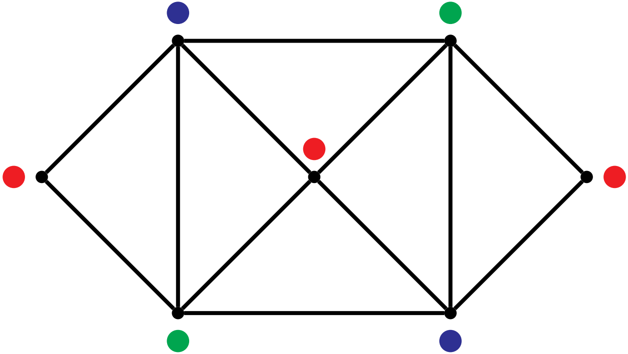
May 6, 2019

General introduction

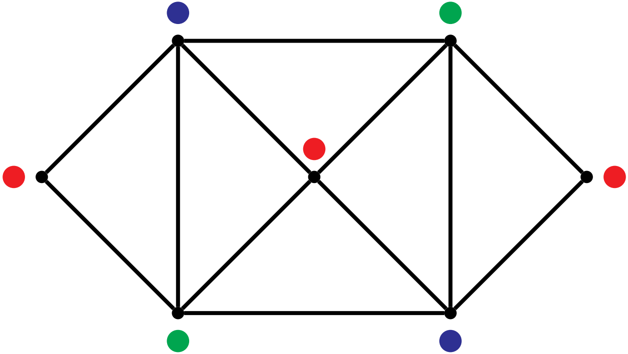
Make adjacent vertices distinguishable?



Make adjacent vertices distinguishable? ⇒ Proper vertex-colouring ☺



Make adjacent vertices distinguishable? $\Rightarrow$ Proper vertex-colouring ☺



⚠ χ might be as high as $\Delta+1$ (Brooks' Theorem)

"Encode" a proper vertex-colouring using few different types of resources?

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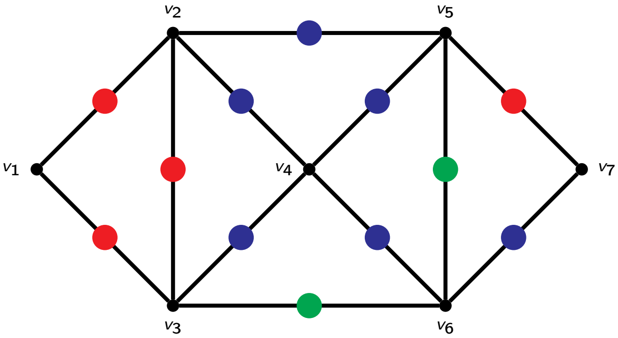
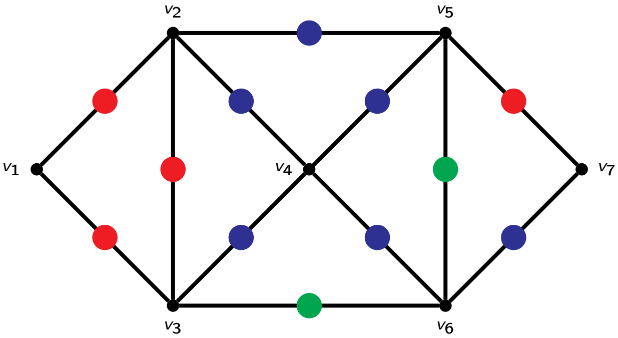


Motivation (cont'd)

"Encode" a proper vertex-colouring using few different types of resources?



$\mathrm{Col}(v_i)$:= Set of colours "incident" to v_i:

$$\mathrm{Col}(v_1) = \{{\color{red}\bullet}\} \quad \mathrm{Col}(v_2) = \{{\color{red}\bullet}, {\color{blue}\bullet}\} \quad \mathrm{Col}(v_3) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$$

$$\mathrm{Col}(v_4) = \{{\color{blue}\bullet}\} \quad \mathrm{Col}(v_5) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_6) = \{{\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_7) = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$$

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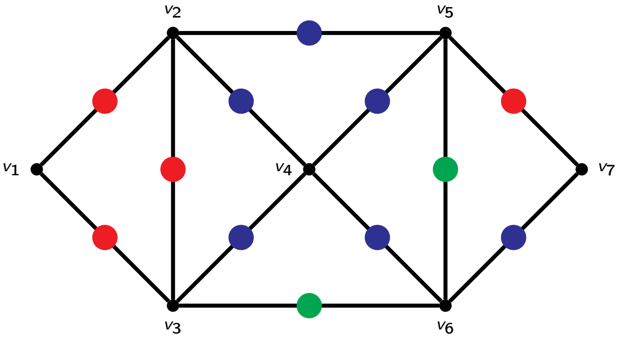

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Neighbours are distinguished!

From edge colours to vertex colours

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- etc.

From edge colours to vertex colours

Many parameters:

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- Edge colour restrictions (properness, etc.)?
- What elements are coloured (edges, vertices, both, etc.)?
- etc.

⇒ Dozens and dozens variants...

A Dynamic Survey of Graph Labeling

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Submitted: September 1, 1996; Accepted: November 14, 1997
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Mathematics Subject Classifications: 05C78

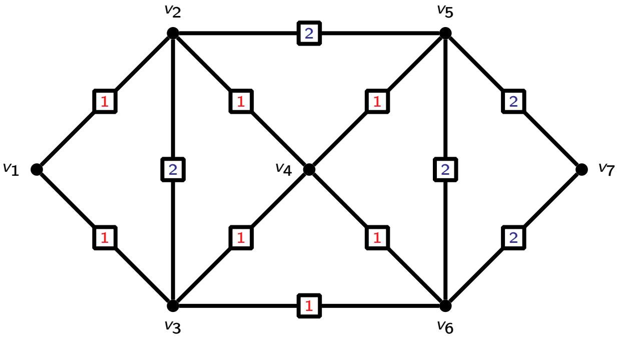
Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 2500 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

1-2-3 Conjecture

– Introduction –

Edge-colours = Edge-weights
$\mathrm{Col}(v_i) = \sigma(v_i) :=$ Sums of weights "incident" to v_i



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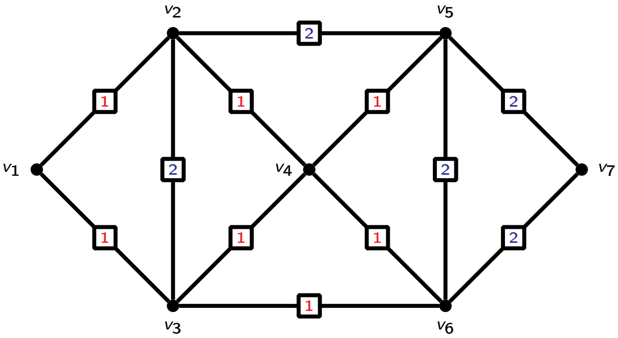

$\sigma(v_1) = 2 \quad \sigma(v_2) = 6 \quad \sigma(v_3) = 5 \quad \sigma(v_4) = 4$
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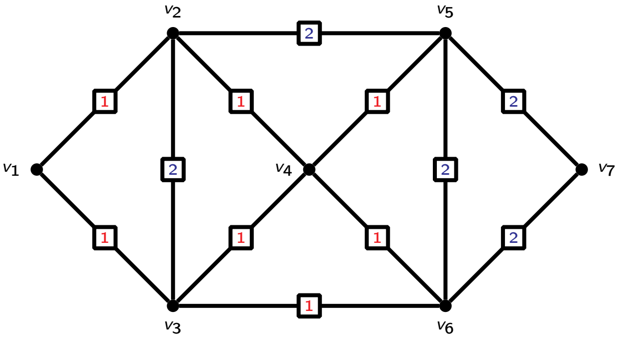

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$\chi^e_\Sigma = 2$ while $\chi = 3$ ☺

1-2-3 Conjecture

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$\chi^e_\Sigma(G)$ = smallest k such that G has n-s-d k-edge-weightings

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Nice graph = no K_2 as a component

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$\chi^e_\Sigma(G)$ = smallest k such that G has n-s-d k-edge-weightings

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Nice graph = no K_2 as a component

1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

For every nice graph G, we have $\chi^e_\Sigma(G) \leq 3$.

Edge weights and vertex colours

Michał Karoński and Tomasz Łuczak

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland
E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
E-mail: a.g.thomason@dpmms.cam.ac.uk

Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?
We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

An additional motivation

This problem is also related to irregular multigraphs

How to Define an Irregular Graph

Gary Chartrand
Paul Erdös
Ortrud R. Oellermann

Q.: regular = same degrees, but irregular = ?
(Note: simple graphs with ≥ 2 vertices of unique degrees do not exist)

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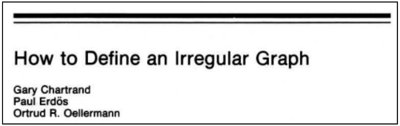
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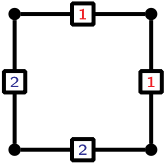
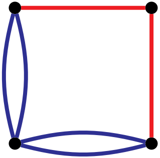
Connection to n-s-d edge-weightings:





$\Rightarrow$ Finding $\chi^e_\Sigma(G)$ $\Leftrightarrow$ Perform this with minimizing maximum edge multiplication

1-2-3 Conjecture
– Some families of graphs –

Theorem

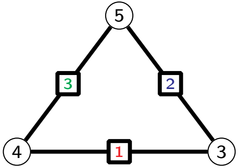
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Make a guess ☺

Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. By induction on n. For $n = 3$:



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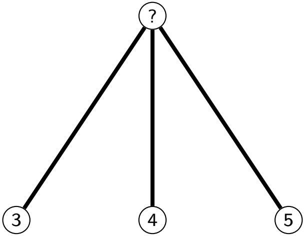
For every $n \geq 3$, we have $\chi^{e}_{\Sigma}(K_n) = 3$.

Proof. $n = 4$:



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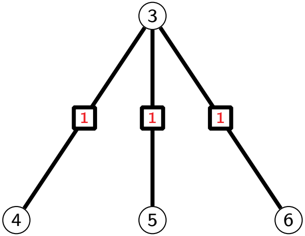

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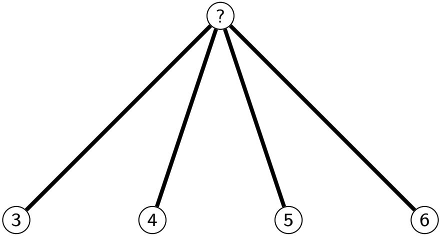
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. $n = 5$:



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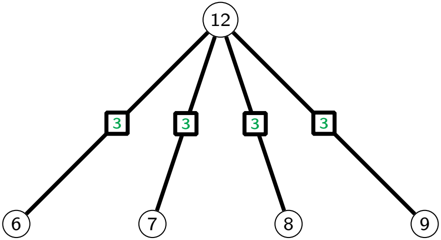

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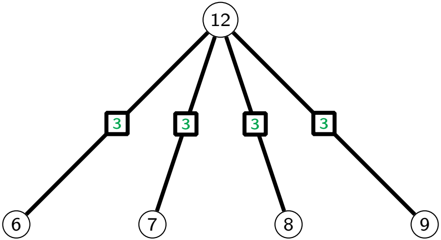

General case: n even $\Rightarrow$ 1's. n odd $\Rightarrow$ 3's.

∎

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Any idea ☺ ?

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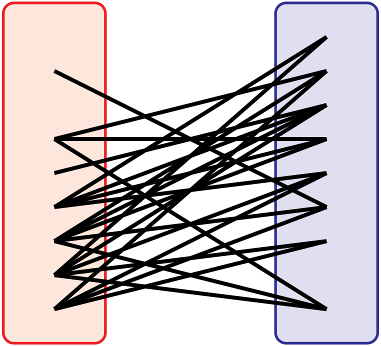
Proof. Bipartition (A, B)



Aim: 3-edge-weighting where $\sigma(A) \equiv 1, 2 \pmod 3$ and $\sigma(B) \equiv 0 \pmod 3$

$\Leftrightarrow$ $\{0, 1, 2\}$-edge-weighting with the same properties

Theorem

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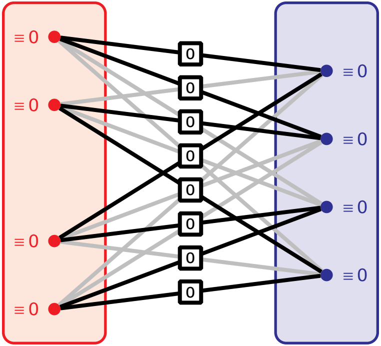
Proof. Assume $|A|$ is even. Start with weights 0. Second condition fulfilled by B.



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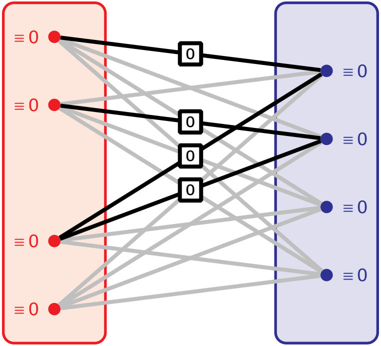
For every nice bipartite graph G, we have $\chi^e_\Sigma(G) \leq 3$.

Proof. Pick a path from A to A with new ends, and apply $+1, -1, \ldots$ along



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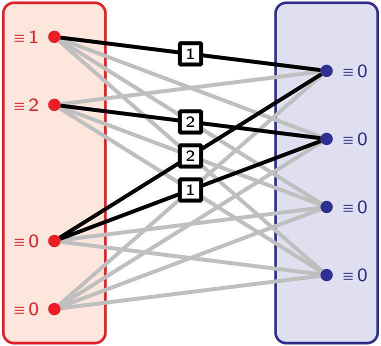

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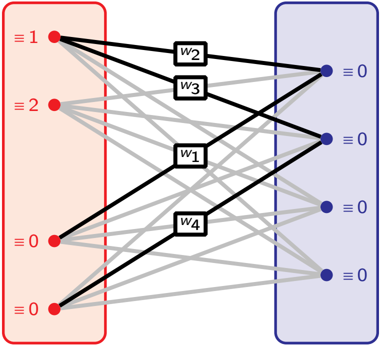

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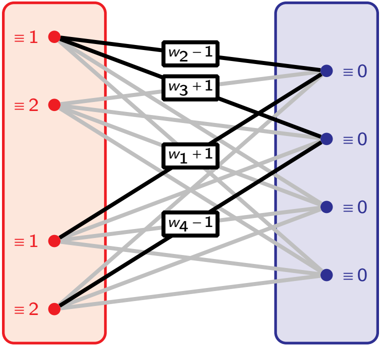
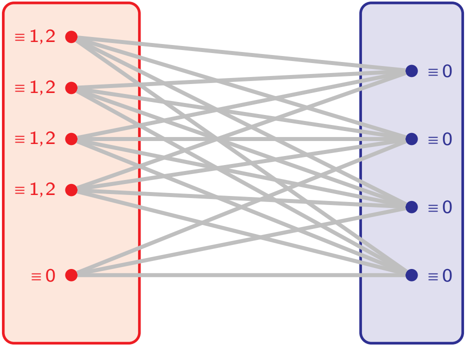


Theorem

For every nice bipartite graph G, we have $\chi^e_\Sigma(G) \leq 3$.

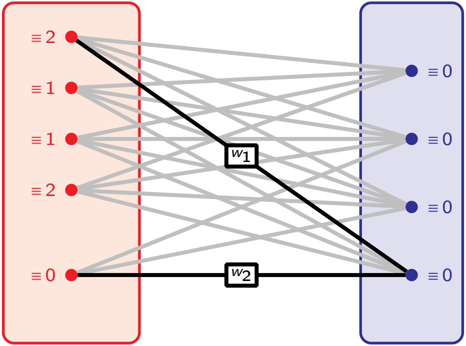
Proof. If $|A|$ and $|B|$ are odd ☹ ... but can reach:



Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Proof. Eventually apply $+1, -1, \ldots$ or conversely towards another vertex in A



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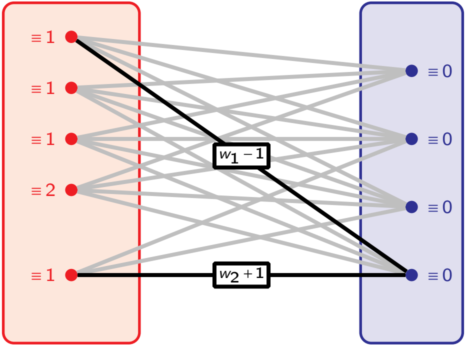

On that proof...

- Proof applies to 3-chromatic graphs with partite sets A, B, C:
  - Use weights $0, 1, 2$
  - Aim $\sigma(A) \equiv 0 \pmod 3$, $\sigma(B) \equiv 1 \pmod 3$, $\sigma(C) \equiv 2 \pmod 3$

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- More generally, k-chromatic graphs, $k \geq 3$ odd, with partite sets $S_0, \dots, S_{k-1}$:
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  - Aim $\sigma(S_i) \equiv i \pmod k$ for $i = 0, \dots, k-1$
- k-chromatic graphs, $k \geq 4$ even, same trick as bipartite graphs

- In general, using $\{1,2,3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi^e_\Sigma \leq 2$ is NP-complete [Dudek, Wajc, 2011]

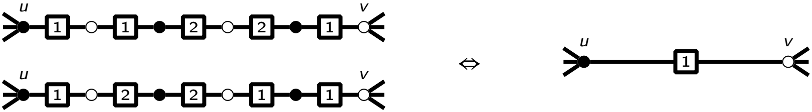
More on the 1-2-3 Conjecture

- In general, using $\{1,2,3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi^{e}_{\Sigma} \leq 2$ is NP-complete [Dudek, Wajc, 2011]

- **Q.:** Is this true for bipartite graphs?
  A.: $\chi^{e}_{\Sigma}(\text{Bipartite}) = 3$: *odd multicacti* [Thomassen, Wu, Zhang, 2016]

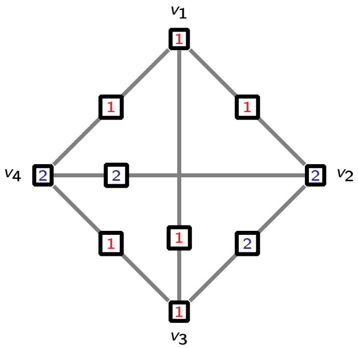
> These graphs can also be described in another way as follows. Take a collection of simple cycles each of length 2 modulo 4 and each with edges colored alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity ≥ 1. The graph with one edge and two vertices is also called an odd multi-cactus.

Intuition: Essentially, with $\{1,2\}$, paths of length $\equiv 1 \pmod 4$ act as edges:



1-2-3 Conjecture
– Best bound –

Best bound on χ^e_Σ obtained from one for a total variant of the problem



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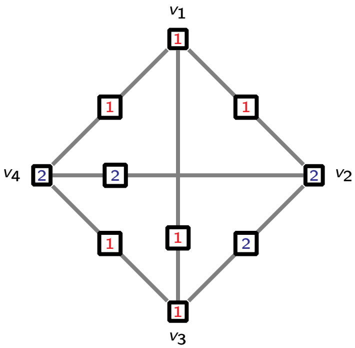

$\sigma(v_1) = 4 \qquad \sigma(v_2) = 7 \qquad \sigma(v_3) = 5 \qquad \sigma(v_4) = 6$

($\sim$ adding a loop at each vertex)
$\chi^t_\Sigma(G)$ = smallest k such that G has n-s-d k-total-weightings

Remarks:

- $\chi_{\Sigma}^{t}(G)$ defined for all G
- $\chi_{\Sigma}^{t}(G) \leq \chi_{\Sigma}^{e}(G)$ for every G

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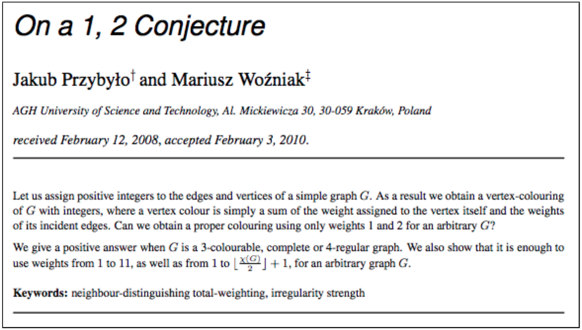

On a 1, 2 Conjecture

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received February 12, 2008, accepted February 3, 2010.

Let us assign positive integers to the edges and vertices of a simple graph G. As a result we obtain a vertex-colouring of G with integers, where a vertex colour is simply a sum of the weight assigned to the vertex itself and the weights of its incident edges. Can we obtain a proper colouring using only weights 1 and 2 for an arbitrary G?

We give a positive answer when G is a 3-colourable, complete or 4-regular graph. We also show that it is enough to use weights from 1 to 11, as well as from 1 to $\lfloor \frac{\chi(G)}{2} \rfloor + 1$, for an arbitrary graph G.

Keywords: neighbour-distinguishing total-weighting, irregularity strength

1-2 Conjecture [Przybyło, Woźniak, 2010]

For every graph G, we have $\chi^t_\Sigma(G) \leq 2$.

Theorem [Kalkowski, 2009]

For every graph G, we have $\chi^t_\Sigma(G) \leq 3$.

Proof. Essentially, a clever induction:

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- Start from all edges at 2, all vertices at 1
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  - Choose a set $\{\phi(v_i), \phi(v_i)+1\}$ of possible sums
    ($\phi(v_i)+1$ = eventual sum, $\phi(v_i)$ the only allowed different sum)
    ⚠ Make sure that $\phi(v_i) \neq \phi(v_j)$ for every backward edge!

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    ⚠ Make sure that $\phi(v_i) \neq \phi(v_j)$ for every backward edge!
  - Make "valid" weight changes backwards so that $\sigma(v_i) \in \{\phi(v_i), \phi(v_i)+1\}$

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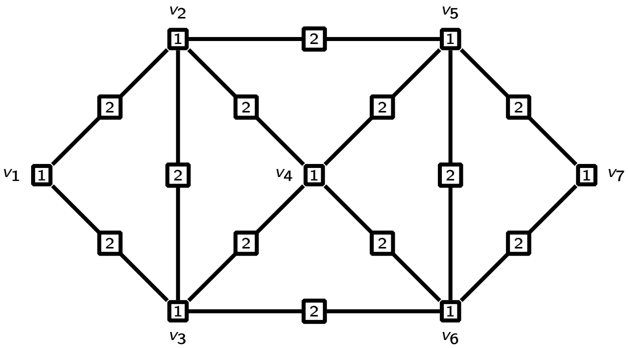
    ⚠ Make sure that $\phi(v_i) \neq \phi(v_j)$ for every backward edge!
  - Make "valid" weight changes backwards so that $\sigma(v_i) \in \{\phi(v_i), \phi(v_i)+1\}$
- Eventually, do +1 on every vertex weight where $\sigma(v_i) = \phi(v_i)$

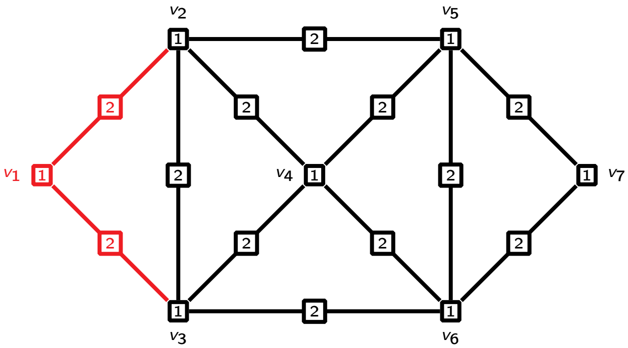
Note: Actually, only $1, 2$ are used as vertex weights

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



Kalkowski's Algorithm: Dealing with v_1

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



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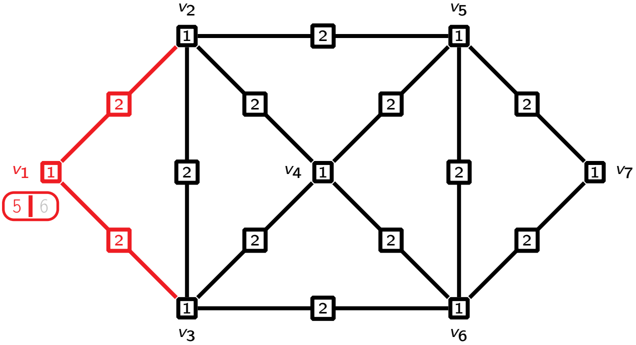

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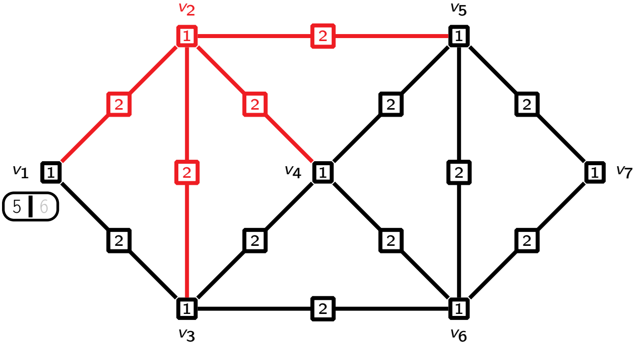

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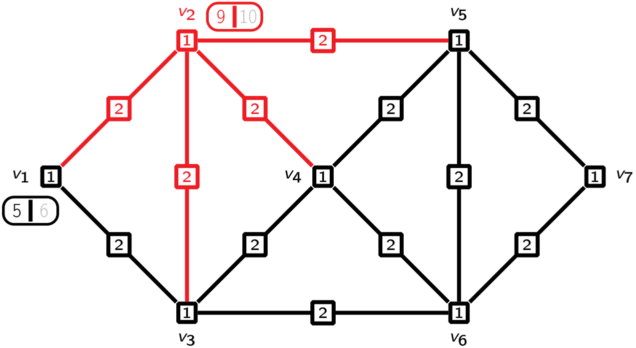
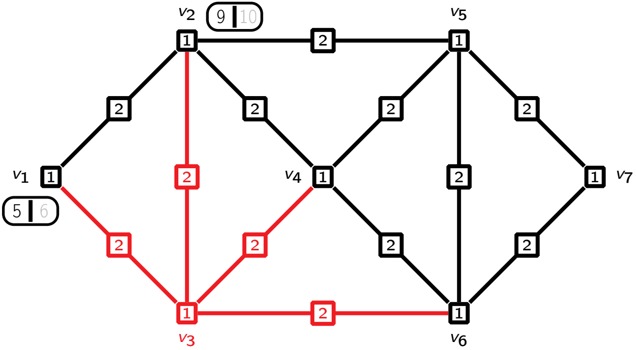
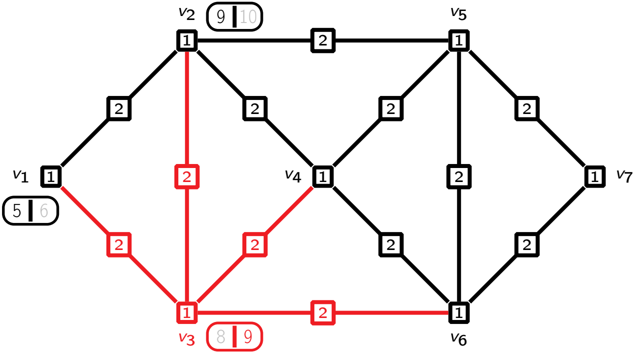


Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



Kalkowski's Algorithm: Dealing with v_3

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



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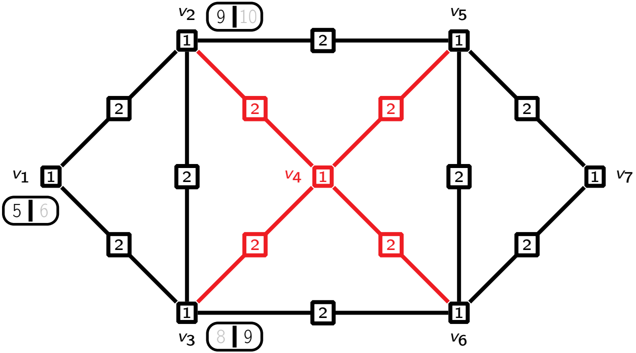

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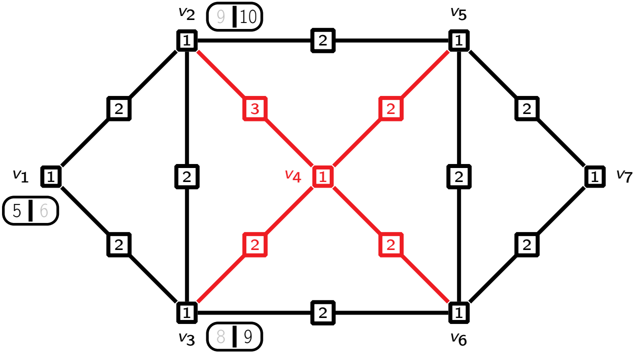

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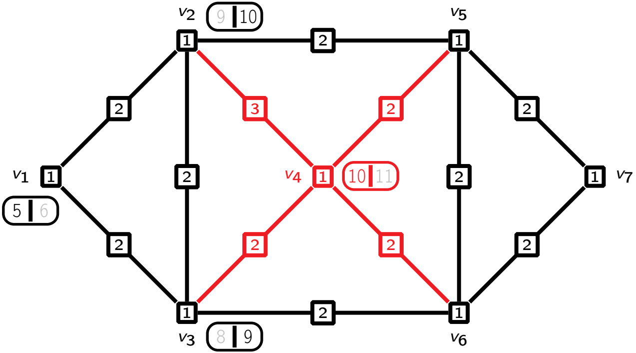

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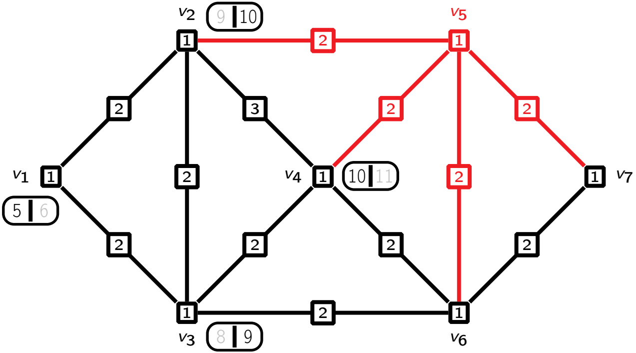

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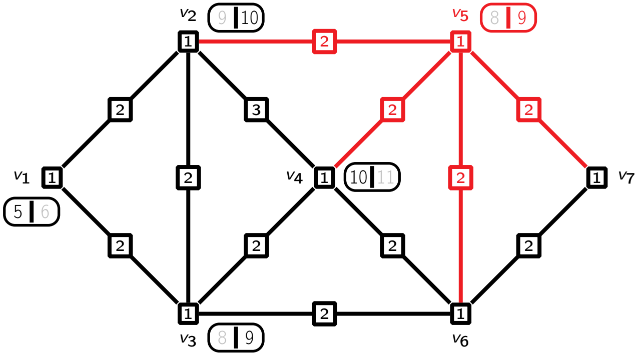
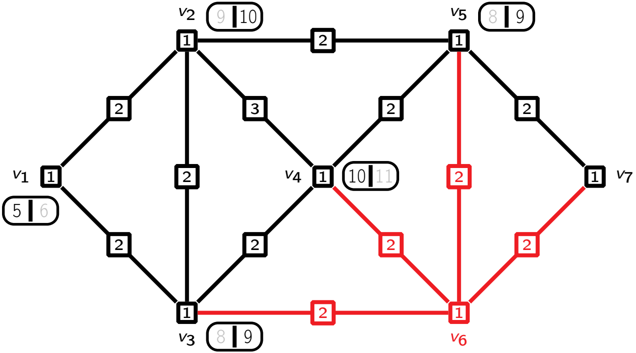
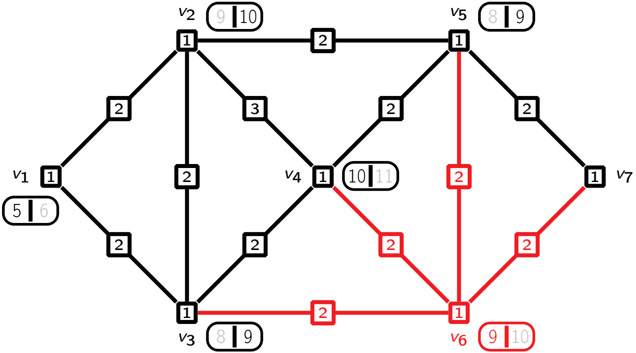


Kalkowski's Algorithm: Dealing with v_6

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



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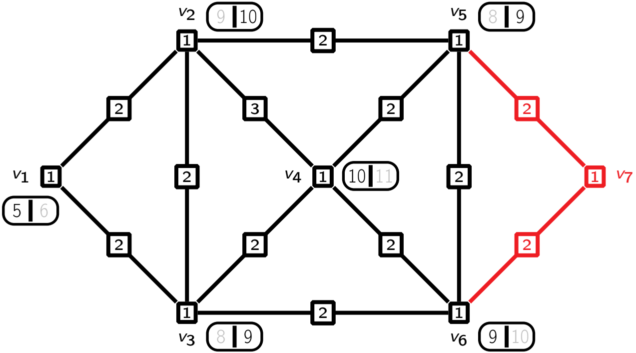

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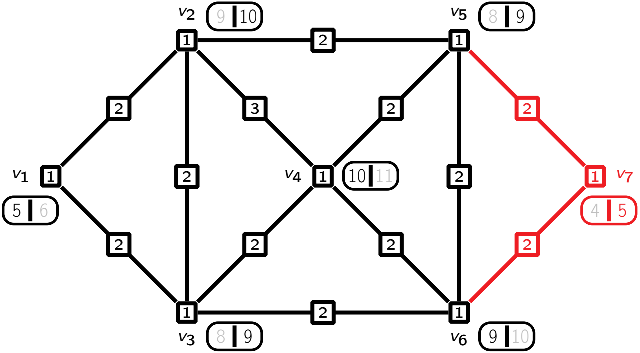
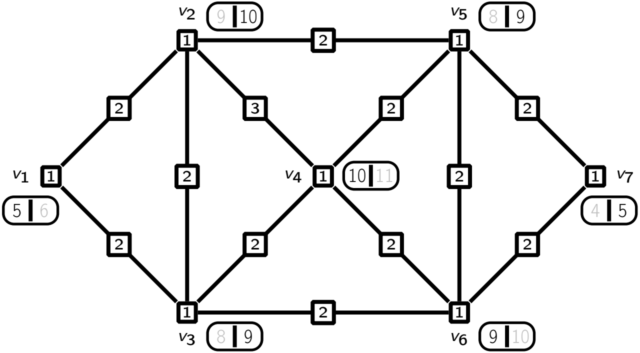


Kalkowski's Algorithm: Final adjustments



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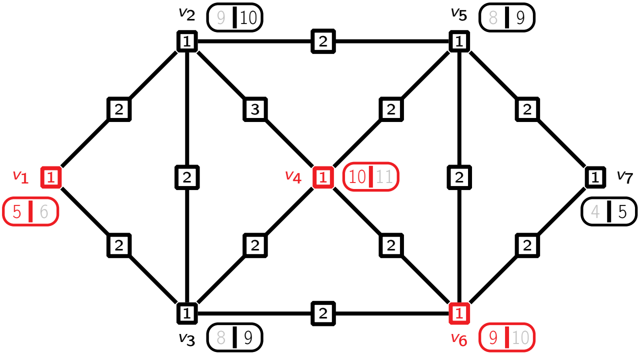

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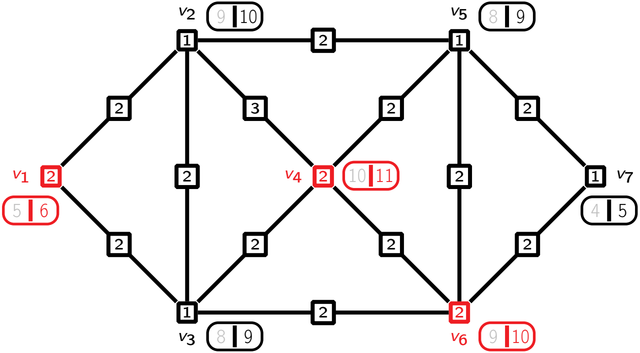
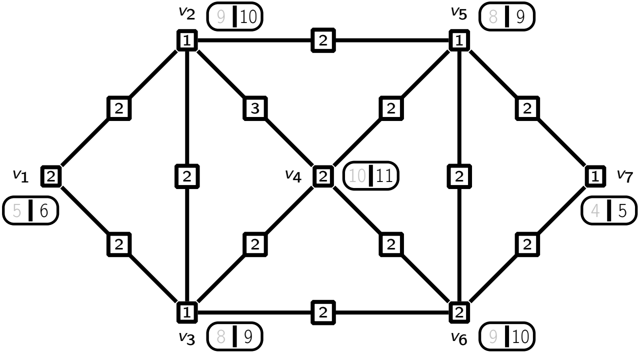


Kalkowski's Algorithm: Final picture



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  - ⚠ Valid changes backwards are trickier

1-2-3 Conjecture
– Open questions –

Some open problems/questions

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  - Done for 4-edge-connected 4-chromatic graphs [Wu, Zhang, Zhu, 2017]

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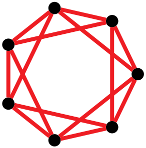
- Prove/Disprove the 1-2 Conjecture
- List variants?
  - Every graph is (2,3)-choosable [Wong, Zhu, 2016]
  - No constant bound for the edge version ☹

Locally irregular decompositions

– Introduction –

Locally irregular = Every two adjacent vertices have distinct degrees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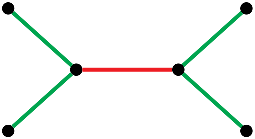
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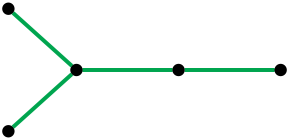
✓

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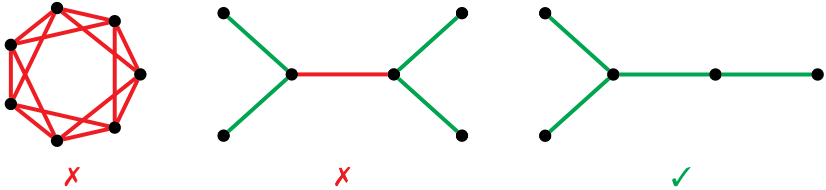

Decomposition of G = Partition $E_1, ..., E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, locally irregular edge-colouring)

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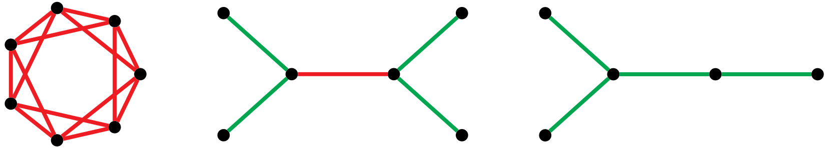

✗ ✗ ✓

Decomposition of G = Partition $E_1, \ldots, E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs
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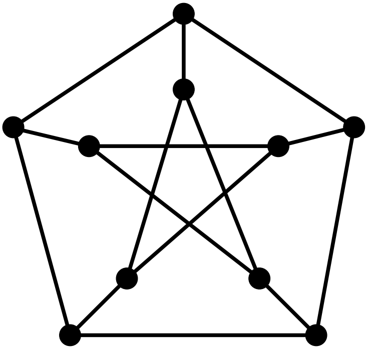
$\chi'_{\mathrm{irr}}(G)$ = Smallest $k \geq 1$ s.t. G has locally irregular k-edge-colourings
G decomposable = $\chi'_{\mathrm{irr}}(G)$ exists
G exceptional, otherwise

Example



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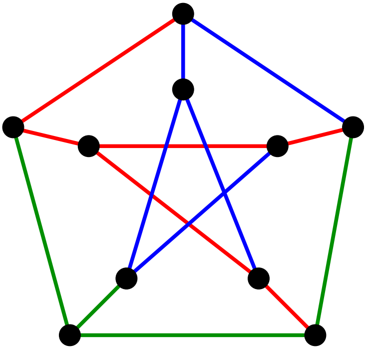

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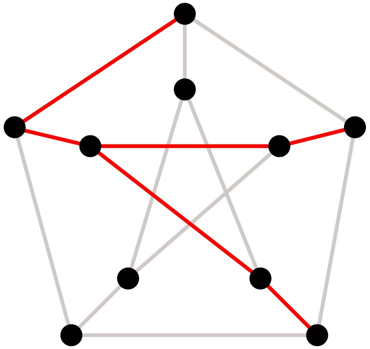

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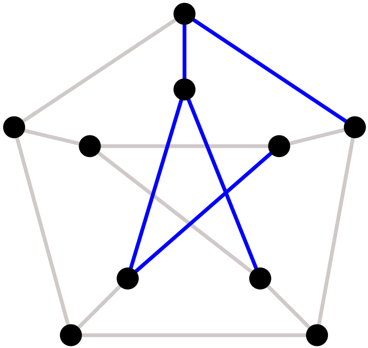

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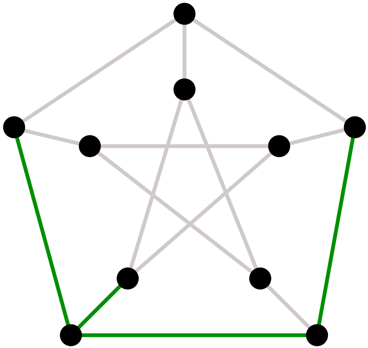

Some motivations

1. Local irregularity = Possible antonym notion to regularity
2. χ'_{irr} = Measure of closeness to irregularity

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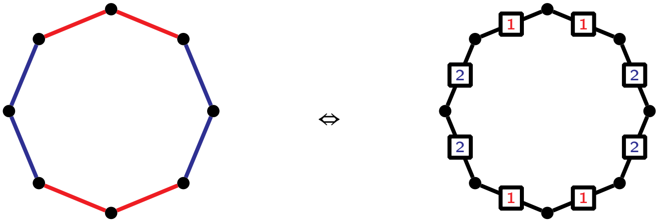
2. χ'_{irr} = Measure of closeness to irregularity
3. Connections and applications to the 1-2-3 Conjecture



In regular graphs, $\chi^e_\Sigma = 2$ if and only if $\chi'_{\text{irr}} = 2$

Exceptional graphs?

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Some obvious ones: odd-length paths and odd-length cycles...

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... but also $\mathscr{T}$:

> Every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. Hence, all exceptional graphs have odd size and a complete characterisation of exceptional graphs was given by Baudon, Bensmail, Przybyło, and Woźniak [1]. To state this characterisation, we first need to define a family $\mathcal{T}$ of graphs. The definition is recursive:
>
> - The triangle K_3 belongs to $\mathcal{T}$.
> - Every other graph in $\mathcal{T}$ can be constructed by (1) taking an auxiliary graph F being either a path of even length or a path of odd length with a triangle glued to one of its ends, then (2) choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally (3) identifying v with a vertex of degree 1 of F.
>
> In other words, the graphs in $\mathcal{T}$ are obtained by connecting a collection of triangles in a tree-like fashion, using paths with certain lengths, depending on what elements these paths connect. Let us point out that all graphs in $\mathcal{T}$ have maximum degree 3, have odd size, and all of their cycles are triangles.

Theorem [Baudon, B., Przybyło, Woźniak, 2015]

Exceptional graphs are exactly these three classes of graphs.

How large can χ'_{irr} be?

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Conjecture [Baudon, B., Przybyło, Woźniak, 2015]

For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2}, K_n, etc.). Actually, unless P = NP, no "good" characterization of when $\chi'_{\mathrm{irr}}(G) \leq 2$ [Baudon, B., Sopena, 2015].

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Conjecture verified for:

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- graphs with $\delta \geq 10^{10}$ [Przybyło, 2016]

Theorem

For every $n \geq 4$, we have $\chi'_{\mathrm{irr}}(K_n) = 3$.

Your turn ☺

Theorem

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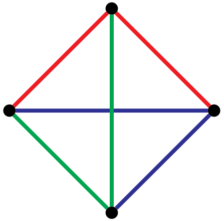
Proof. Quite similar as for the 1-2-3 Conjecture. For $n = 4$:



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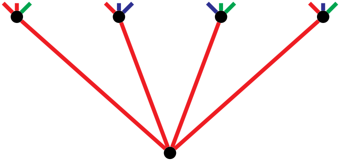

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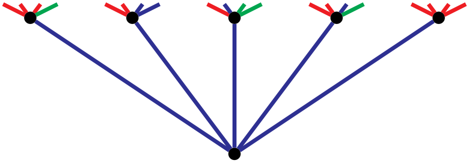
Proof. For $n = 6$:

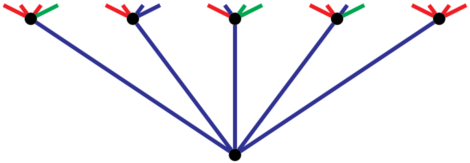


Complete graphs

Theorem

For every $n \geq 4$, we have $\chi'_{\mathrm{irr}}(K_n) = 3$.

Proof. For $n = 6$:

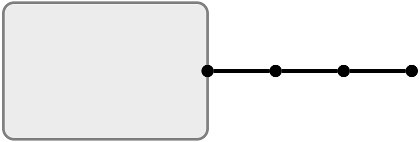


General case: n even ⇒ /'s. n odd ⇒ /'s. ■

Theorem

For every decomposable tree T, we have $\chi'_{\text{irr}}(T) \leq 3$.

Proof. For instance, by induction. If a pendant path of length ≥ 3:



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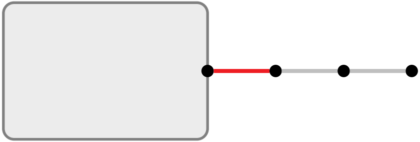

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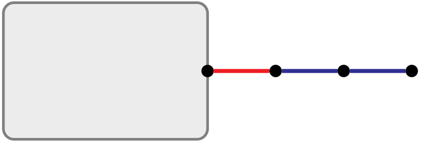

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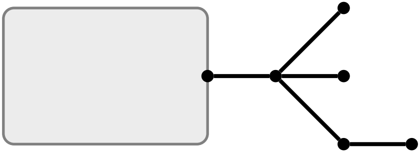

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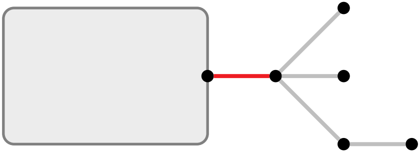

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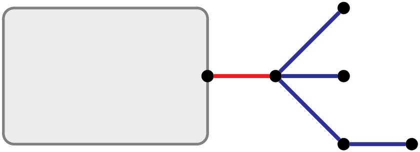

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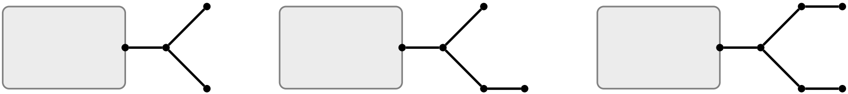

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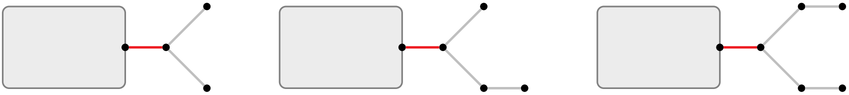

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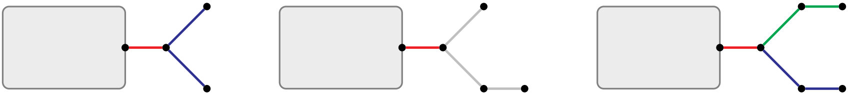

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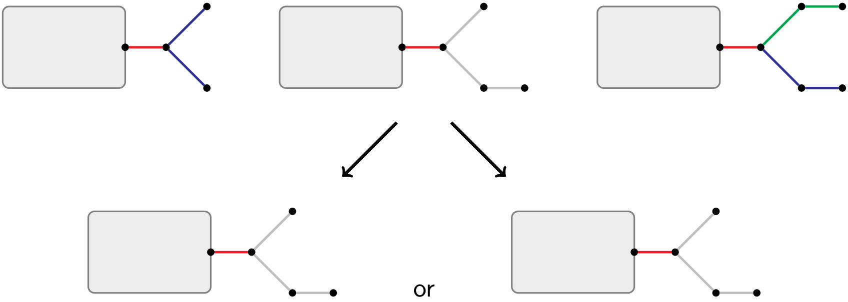

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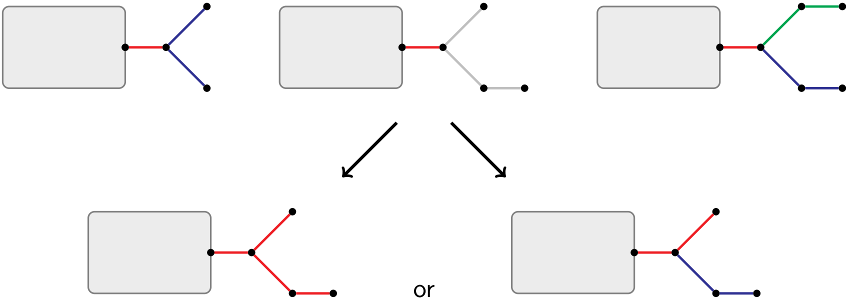

∎

Theorem [B., Merker, Thomassen, 2016]

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, \ldots, G_k$ of G s.t.

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Recent result

- Lužar, Przybyło and Soták proved $\chi'_{\mathrm{irr}}(\mathrm{decomposable}, \mathrm{bipartite}) \leq 6$
- $\Rightarrow$ $\chi'_{\mathrm{irr}}(\mathrm{decomposable}) \leq 220$!
- (just plug new result in previous approach)

Locally irregular decompositions

– Open questions –

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- Better general bounds via different approaches?

A generalization

– Introduction –

Coloured weights and sums

Each edge → Coloured weight (α, β) w/ colour α and value β
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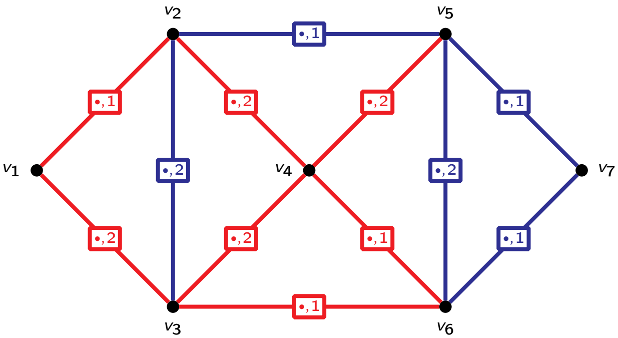

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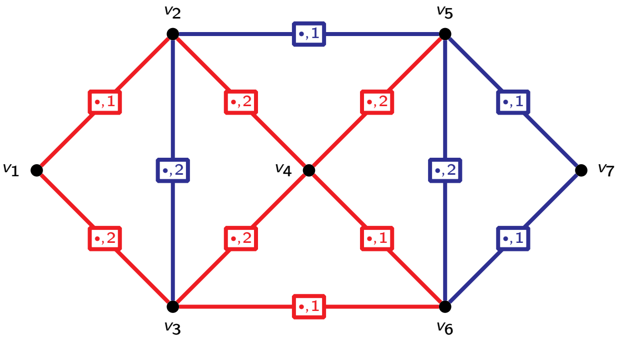

$\sigma_{\color{red}\bullet}(v_1) = 3,\ \sigma_{\color{blue}\bullet}(v_1) = 0$ $\quad$ $\sigma_{\color{red}\bullet}(v_2) = 3,\ \sigma_{\color{blue}\bullet}(v_2) = 3$ $\quad$ $\sigma_{\color{red}\bullet}(v_3) = 5,\ \sigma_{\color{blue}\bullet}(v_3) = 2$

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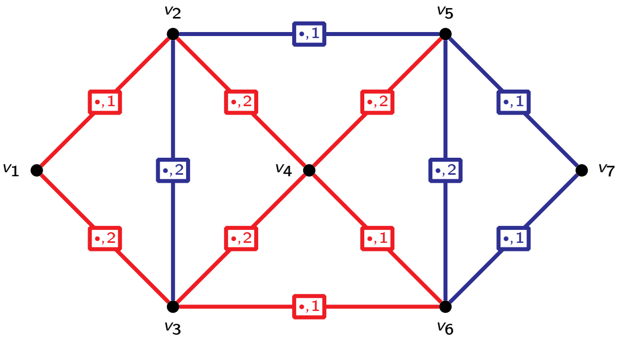

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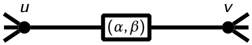
When are adjacent vertices considered distinguished?

Colours $\in \{1, \ldots, \alpha\}$, Weights $\in \{1, \ldots, k\}$

Three more or less strong distinction conditions:



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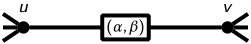

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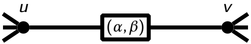

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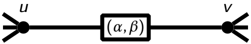

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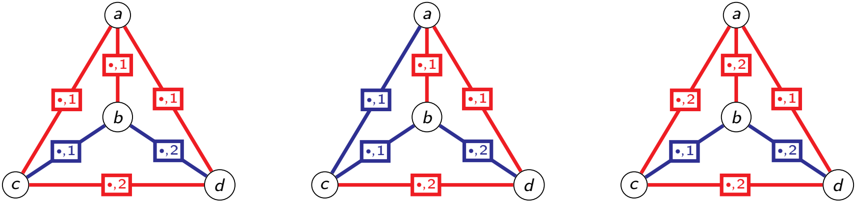
Note: Strong $\Rightarrow$ Standard $\Rightarrow$ Weak; but no converse is true:



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Also: For $k = 1$:

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  - 1-2-3 Conjecture = Are all nice graphs strongly (1,3)-colourable?
  - They are strongly (1,5)-colourable

Also: For $k = 1$:

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  - L.I. Conjecture = Are all decomposable graphs standardly (3,1)-colourable?
  - They are all (220,1)-colourable

Remarks and connections

(ℓ,k)-colouring: colour pool $\{1,...,\ell\}$, weight pool $\{1,...,k\}$
weak/standard/strong colouring: each edge fulfils the corresponding condition

Remarks: For $\ell = 1$:

- weak = standard = strong
- strong $(1,k)$-colouring = n-s-d k-edge-weighting
- Hence:
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  - They are all (220,1)-colourable
- also, weak $(\ell,1)$-colouring = ℓ-edge-colouring distinguishing by *multisets*
- Hence:
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Playing with at least two colours and at least two weights?

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Conjecture [Baudon, B., Davot, Hocquard, Przybyło, Senhaji, Sopena, Woźniak, 2019]

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Recall: "Strong Conjecture" ⇒ "Standard Conjecture" ⇒ "Weak Conjecture"

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  $\Rightarrow$ Proof reduces to odd multicacti

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Standard Conjecture

Nicer graphs decompose into two graphs fulfilling the 1-2-3 Conjecture.

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  ⇒ Via basic induction, decomposition into two nice forests
- Subcubic graphs
  ⇒ Via the previous result + induction
- 9-colourable graphs
  ⇒ Decompositions into two nice 3-colourable graphs

Watch out: When using induction, ⚠ bad components!

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Bound towards $(3, 1)$**-colourability:**

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Recall: Results towards the Strong or Standard Conjecture apply

A generalization

– Open questions –

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Thank you for your attention!