Going beyond the 1-2-3 Conjecture

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Abstract

The so-called 1-2-3 Conjecture, raised by Karonski, Łuczak, and Thomason in 2004, asks whether all connected graphs different from K2 can have their edges labelled with 1, 2, and 3 so that no two adjacent vertices are incident to the same sum of labels [KLT04]. Since its introduction, the 1-2-3 Conjecture has been attracting increasing attention, resulting in many interesting works and results. In particular, an aspect making the field of interest is the variety and diversity of aspects and types of questions that can be considered. As an illustration, let us mention that various structural and algorithmic concerns have been investigated to date, involving arguments of sometimes quite different natures (algebraic, probabilistic, algorithmic, etc.). There are even quite a lot of seemingly unrelated connections between the 1-2-3 Conjecture and other notions of graph theory, such as stable sets, flows, maximum cuts, proper vertex-colourings, and so on. See the survey [Sea12] by Seamone for more details.

There have been quite some works on the topic over the recent years, leading to many results of interest. In particular, Vučković recently proved the multiset version of the 1-2-3 Conjecture [Vuc18], before Bensmail, Hocquard, Lajou, and Sopena proved its product version [BHLS23]. Very recently, in early 2023, Keusch proposed a full solution to the whole 1-2-3 Conjecture [Keu23+].

Objectives of the internship

Despite Keusch's proof of the 1-2-3 Conjecture, the field remains full of related interesting open questions and problems. The main goal of the internship will thus consist in understanding the full details of Keusch's proof (which mainly rely on an innovative use of flows) and investigating whether these can bring anything new for any variant of the 1-2-3 Conjecture. As examples of such variants, let us mention its total variant (where both vertices and edges are to be labelled with labels 1 and 2), its tree variant (where edges must be labelled with labels 1 and 2 so that the resulting sums induce forests), its proper variants (where labels assigned to edges must form a proper edge-colouring), its equitable variants (where labels must be assigned in a somewhat equitable way), some of its generalisations to structures more general than graphs (such as

digraphs, signed graphs, hypergraphs, etc.), and some of its optimisation variants (where a specific metric, such as the sum of assigned labels, the maximum sum resulting, etc., must also be minimised/maximised).

This apart, as mentioned earlier quite some questions and variants related to the 1-2-3 Conjecture remain open to date (see [Sea12]), so, more generally speaking, side aspects of the internship will also be to get any interesting progress towards any of these.

Prerequisites

Applicants are expected to have good knowledge, experience, and interest for most of the fields involved, in particular for chromatic theory, structural graph theory, and algorithmic theory. Depending on the questions to be considered, coding skills might also be useful, *e.g.* for exhibiting graphs with certain desired properties.

Additional information

The selected candidate will be supervised by three researchers (including a Ph.D. student working on the topic of interest) to guarantee a diligent supervision throughout the period. The candidate is also expected to take part to the activities of the Graphs and Optimisation research group, and thus to interact with its several members, in particular at the occasion of the weekly team seminar. The interested candidate will also have the opportunity to attend any of the several seminars held at LaBRI on a weekly basis.

As mentioned earlier, many aspects of the field could be worth studying. In particular, depending on the candidate's interests and the achievements reached, the internship could give birth to a Ph.D. thesis proposal.

References

[BHLS23] J. Bensmail, H. Hocquard, D. Lajou, É. Sopena. A proof of the Multiplicative 1-2-3 Conjecture. Combinatorica, 43:37-55, 2023.

[KLT04] M. Karonski, T. Łuczak, A. Thomason. Edge weights and vertex colours. Journal of Combinatorial Theory, Series B, 91:151–157, 2004.

[Keu23+] R. Keusch. A Solution to the 1-2-3 Conjecture. Preprint, 2023. Available online at http://arxiv.org/abs/2303.02611.

[Sea12] B. Seamone. The 1-2-3 Conjecture and related problems: a survey. Preprint, 2012. Available online at http://arxiv.org/abs/1211.5122.

[Vuc18] B. Vučković. Multi-set neighbor distinguishing 3-edge coloring. Discrete Mathematics, 341:820-824, 2018.