Partitions and decompositions of graphs

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Foreword

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- *Vertex-partition* into *connected subgraphs* with *prescribed orders*

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- *Vertex-partition* into *connected subgraphs* with *prescribed orders*
- *Introduction* of *irregularity* via an *edge-colouring*

In both cases: *Algorithmic* and *combinatorial* concerns

- First problem -

Vertex-partitioning graphs into connected subgraphs

Kalinowski Marczyk Pilśniak Przybyło Woźniak

Baudon Foucaud Sopena

A few terminology

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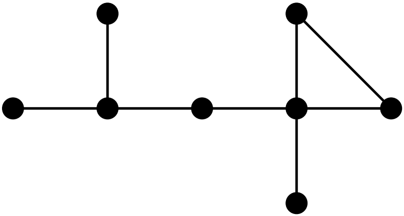

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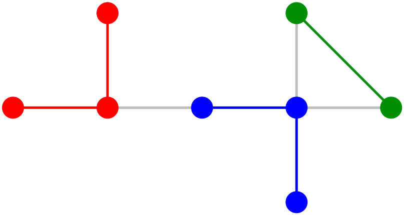

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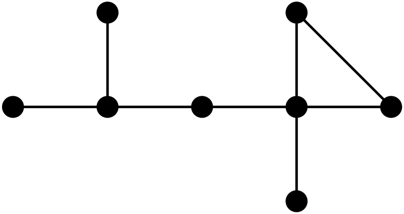

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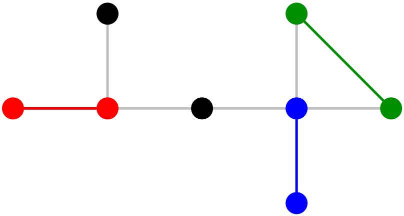

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Our considerations

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Graphs in which all sequences are realizable

Notion of “best” partitionable graph [Barth, Baudon, Puech, 2002]

Arbitrarily partitionable (AP) graph

G is *arbitrarily partitionable* (AP) if all sequences are realizable in G.

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Theorem [Baudon, Foucaud, Przybyło, Woźniak, 2014]

Removing at least two vertices from an AP graph may result in infinitely many components, but their orders follow an exponential growth.

Hardness of realizing sequences in graphs

REALIZATION

Input: a graph G and a sequence π.
Question: is π realizable in G?

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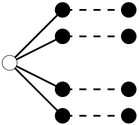
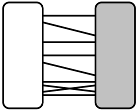
Summarizing theorem

Realization is NP-complete, even when

- $\pi = (3, 3, ..., 3)$ [Dyer, Frieze, 1985],
- G is a subdivided star [B., 2014],
- G is a split graph [Broesma, Kratsch, Woeginger, 2013],
- ...





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NP-completeness of Realization for subdivided stars and split graphs! ...

On polynomial kernels of sequences

Classic idea: reduce the number of sequences to check

(polynomial) Kernel of sequences

A *kernel* for G is a set K of sequences such that

$$G \text{ is AP if and only if } K \text{ is ``realizable'' in } G.$$

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Major open question related to AP graphs:

Conjecture [Barth, Fournier, 2006]

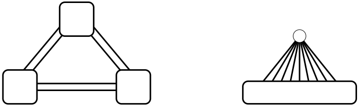
Every graph admits a polynomial kernel.

New positive results on AP Graph

Summarizing theorem [B., 2014]

AP Graph is in

- P when G is a complete multipartite graph,
- NP when G has at least $\left\lceil \frac{|V(G)| - \ln(|V(G)|) - 2}{2} \right\rceil$ universal vertices,
- NP when G is a specific compound graph.



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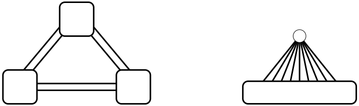

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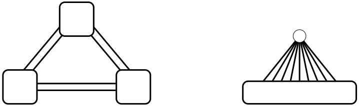

Corollary [Horňák, Marczyk, Schiermeyer, Woźniak, 2012]

Every graph G with at least $\left\lceil \frac{|V(G)|-5}{2} \right\rceil$ universal vertices is AP.

A kernel for graphs with universal vertices

$K_{\mathcal{U}_k}(n) = \{\pi : \text{the greatest element value of } \pi \text{ appears at least } k+1 \text{ times}\}$

Theorem [B., 2014]

$K_{\mathcal{U}_k}(|V(G)|)$ is a kernel for G whenever it has at least k universal vertices.

Proof. Prove that G is AP $\Leftrightarrow$ $K_{\mathcal{U}_k}(|V(G)|)$ is realizable in G

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$\pi' \in K_{\mathcal{U}_k}(|V(G)|)$, which admits a realization in G where the universal vertices are each uniquely included in one big connected subgraph $\rightarrow$ Realization of π in G ■

On the polynomiality of $K_{\mathcal{U}_k}(n)$

Theorem [B., 2014]

$K_{\mathcal{U}_k}(n)$ is a cubic kernel whenever $k \geq \left\lceil \frac{n-\ln(n)-2}{2} \right\rceil$.

Proof. $\pi \in K_{\mathcal{U}_k}(n) = (x \geq k+1 \text{ occurrences of } n_1) + (\text{partition of } n - xn_1)$

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$$\begin{aligned} n - xn_1 &\leq \ln(n) \\ n - (k+1)2 &\leq \ln(n) \\ &\dots \\ k &\geq \left\lceil \frac{n - \ln(n) - 2}{2} \right\rceil \end{aligned}$$

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? Consider large value of graph invariants (e.g. density, average degree, etc.)
? Other graph classes (e.g. triangulated plane graphs)

- Second problem -

Introducing irregularity in graphs via an edge-colouring

Przybyło Stevens Woźniak
Baudon Renault Sopena

Regularity VS Irregularity

Regular graph: all vertices have the *same* degree

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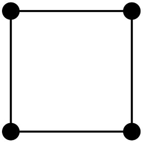
G simple graph, at least two vertices: cannot be totally irregular!

Question [Chartrand et al., 1988]

What is the least integer $x \geq 2$ such that G can be turned into a totally irregular *multigraph* by multiplying each of its edges at most x times?



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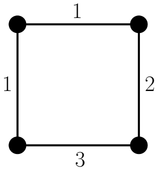

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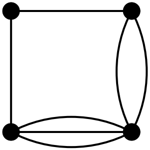

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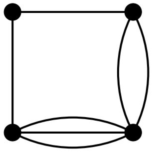

$x \leq |V(G)|$ [Nierhoff, 2000]

Locally irregular graphs

Another definition of irregularity for simple graphs [Alavi *et al.*, 1987]

Locally irregular graph

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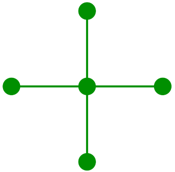

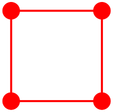

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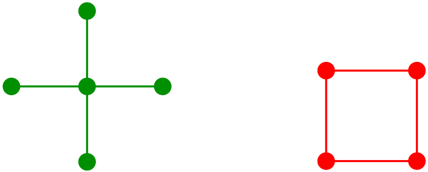

Turning G into a locally irregular multigraph?

1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

$x \leq 3$.

$x \leq 5$ [Kalkowski, Karoński, Pfender, 2010]

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Our approach

Decomposing G into edge-disjoint locally irregular subgraphs

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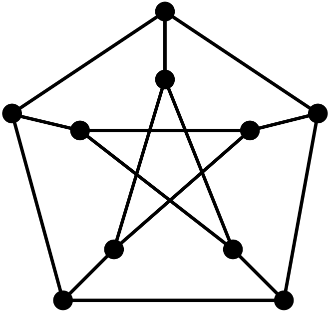
An edge-colouring is *locally irregular* if every colour class induces a locally irregular subgraph.



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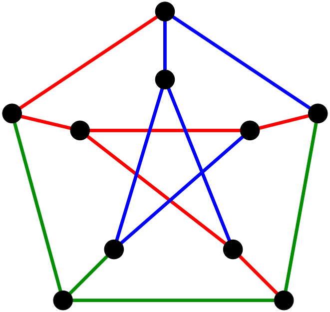

Our approach

Decomposing G into edge-disjoint locally irregular subgraphs

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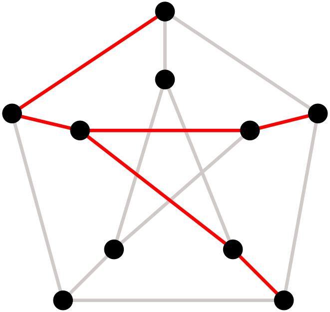

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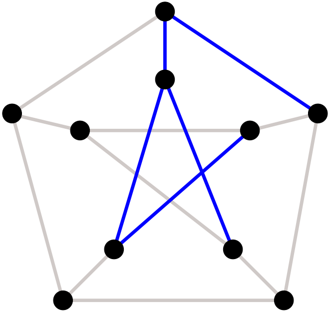

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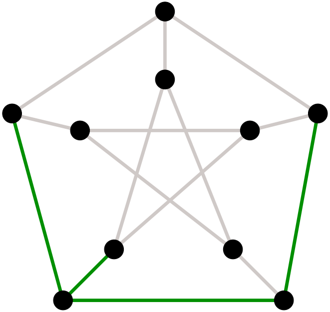

Main conjecture

Irregular chromatic index

The *irregular chromatic index* of G, denoted $\chi'_{irr}(G)$, is

$$\min\{k : G \text{ admits a locally irregular } k\text{-edge-colouring}\}.$$

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An *exception* is a graph with infinite irregular chromatic index. A *colourable* graph is a graph which is not an exception.

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An *exception* is a graph with infinite irregular chromatic index. A *colourable* graph is a graph which is not an exception.

Conjecture [Baudon, B., Przybyło, Woźniak, 2013]

If G is colourable, then $\chi'_{irr}(G) \leq 3$.

Exception graphs

Theorem [Baudon, B., Przybyło, Woźniak, 2013]

G is an exception if and only if G is an odd length path or cycle, or a member of $\mathcal{T}$.

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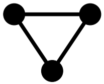

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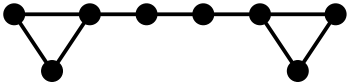

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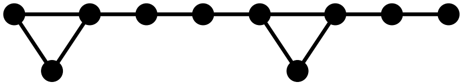

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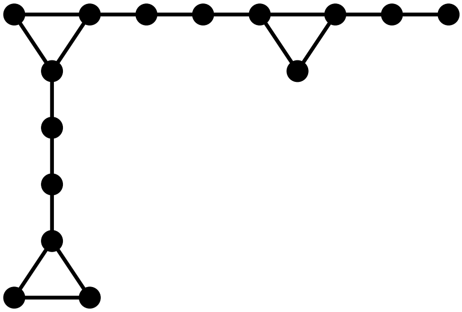

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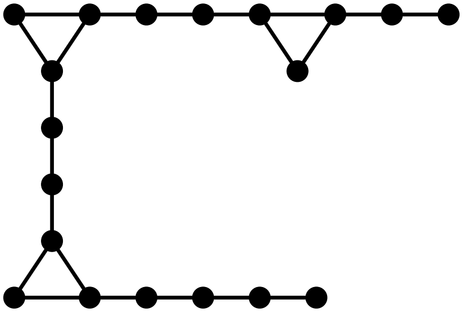

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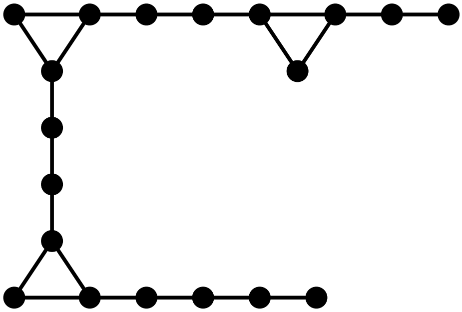

Recognition: polynomial time

On the irregular chromatic index of colourable graphs

Smallest locally irregular non-trivial graph: P_3

Corollary [Baudon, B., Przybyło, Woźniak, 2013]

If G is colourable, then

$$\chi'_{irr}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor.$$

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Summarizing theorem [Baudon, B., Przybyło, Woźniak, 2013]

$\chi'_{irr}(G) \leq 3$ if G is a

- colourable path or cycle,
- particular colourable bipartite graph (including trees),
- complete graph on at least four vertices,
- Cartesian product of graphs verifying the conjecture,
- d-regular graph with $d \geq 10^7$.

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- Cartesian product of graphs with $\chi'_{irr} \leq 3$,
- **d-regular graph with $\mathbf{d \geq 10^7}$.**

Irregular chromatic index of regular graphs

Theorem [Baudon, B., Przybyło, Woźniak, 2013]

If G is d-regular with $d \geq 10^7$, then $\chi'_{irr}(G) \leq 3$.

Proof (sketch). Two steps

Find $E(G) = E_1 \cup E_2 \cup E_3$ yielding three subgraphs G_1, G_2 and G_3 such that

- for every $uv \in E(G)$, we have $d_{G_i}(u) \neq d_{G_j}(v)$ for every $i \neq j$
- each vertex u has degree "almost" $d_G(u)/3$ in G_1, G_2 and G_3

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Existence of a such degree repartition? $\Rightarrow$ **Lovász Local Lemma!**

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Proof (sketch). Choosing the edges? Use of the following ■

Corollary [Addario-Berry et al., 2007]

Given a positive integer $\lambda \leq \frac{\delta(G)}{6}$ and an assignment

$$t : V \to \{0, 1, ..., \lambda - 1\},$$

there exists a spanning subgraph H of G such that $d_H(v) \in \{\frac{d(v)}{3}, \frac{d(v)}{3} + 1, ..., \frac{2d(v)}{3}\}$, and either $d_H(v) \equiv t(v) \pmod{\lambda}$ or $d_H(v) \equiv t(v) + 1 \pmod{\lambda}$ for every vertex v of G.

Complexity matters

Theorem [Baudon, B., Sopena, 2014]

Determining the irregular chromatic index of a tree T can be done in time $\mathcal{O}(|V(T)|)$.

Theorem [Baudon, B., Sopena, 2014]

Determining whether $\chi'_{irr}(G) \leq 2$ is NP-complete.

Open questions and perspectives

- Characterization of exceptions
- Verification of our conjecture for several classes of graphs
- Positive and negative complexity results

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? Upper bounds of χ'_{irr} involving other graph parameters

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Recent results

What if we allow K_2 in decompositions?

Regular-irregular edge-colouring, Regular-irregular chromatic index

An edge-colouring is *regular-irregular* if every colour class induces a subgraph including regular or locally irregular components. The *regular-irregular chromatic index* of G, denoted $\chi'_{reg-irr}(G)$, is

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Theorem [B., Stevens, 2014]

If G is bipartite, then $\chi'_{reg-irr}(G) \leq 6$.

Proof (sketch). Decomposition into auxiliary structures

$$\text{bipartite} \quad = \quad \text{forest} \quad + \quad \text{Eulerian bipartite}$$

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Theorem [B., Stevens, 2014]

$\chi'_{reg-irr}(G) \leq 6\log(\chi(G))$.

Proof (sketch). *G* decomposes into at most $\log(\chi(G))$ bipartite graphs ∎

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Thank you for your attention