Augmenting matchings in trees, via bounded-length augmentations

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Qinghai Normal University, Xining, China
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Introduction

Cast

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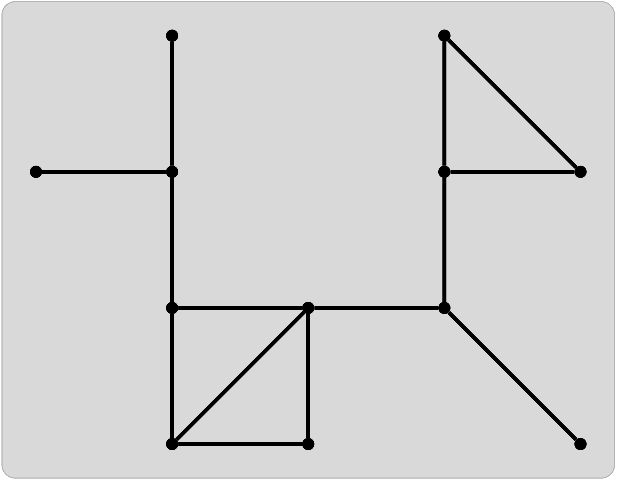
Graph



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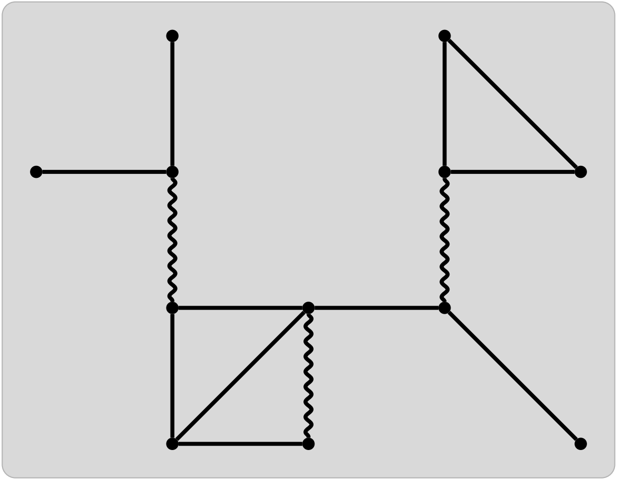
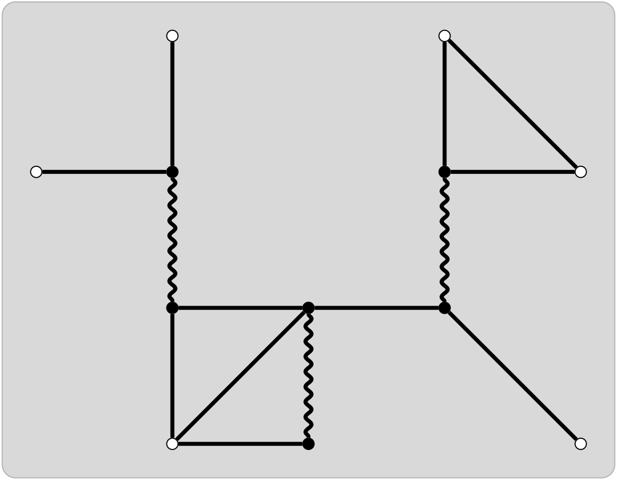
Graph, Matching.



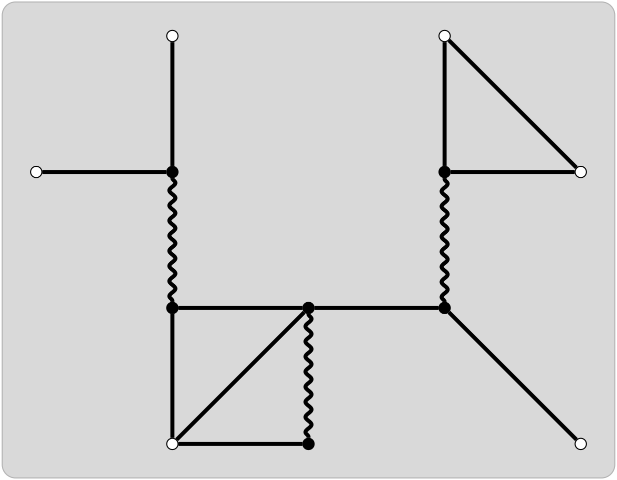
Cast

Graph, Matching.
Exposed vertex, Covered vertex.



Augmenting a matching

Augmenting path, Augmentation.



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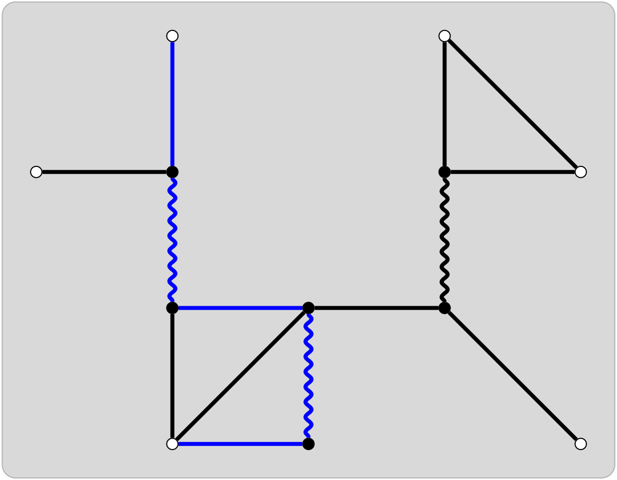

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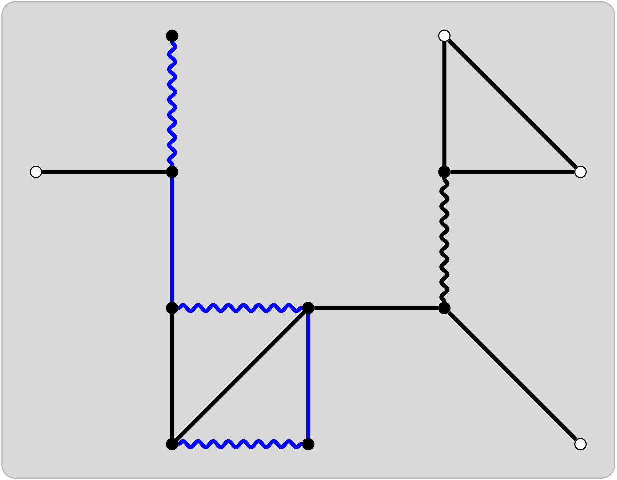

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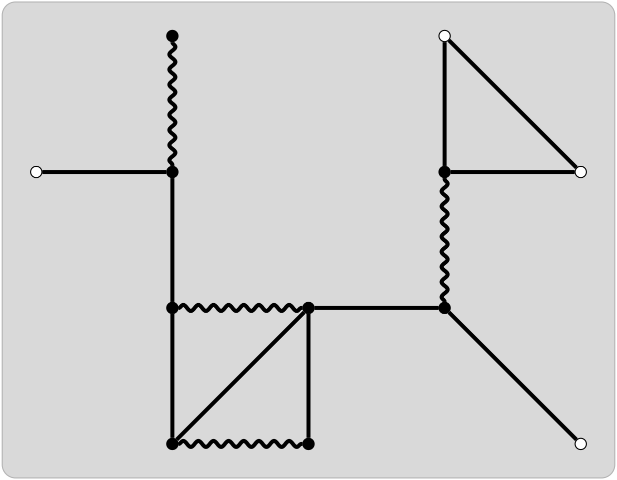

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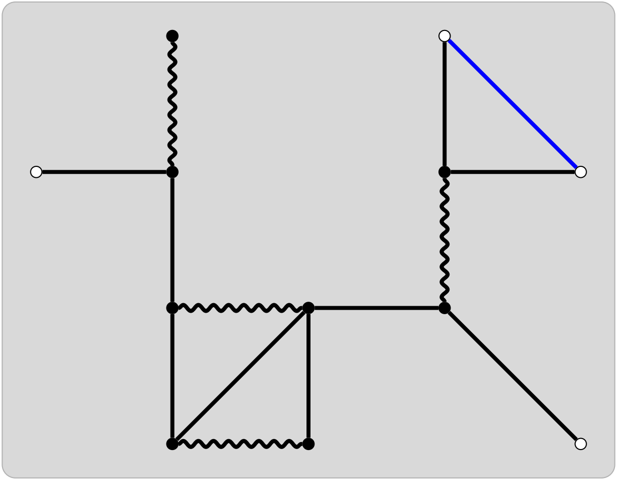

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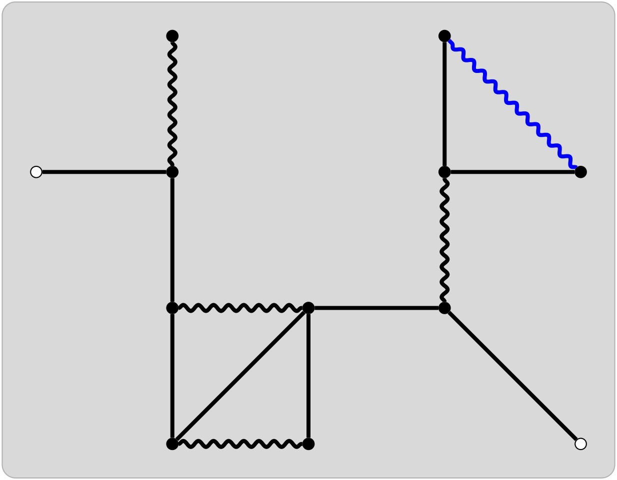

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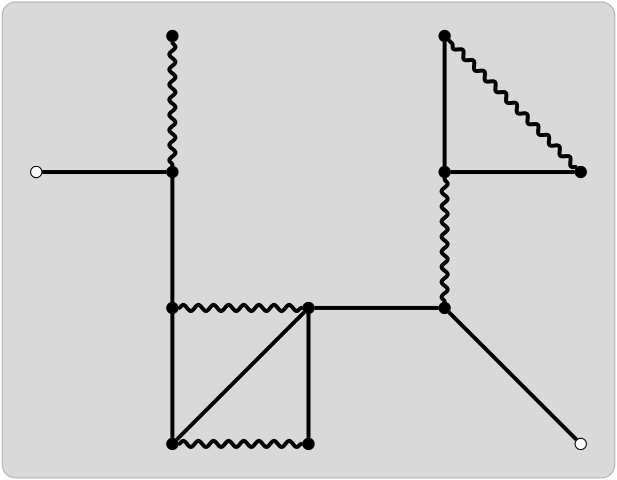

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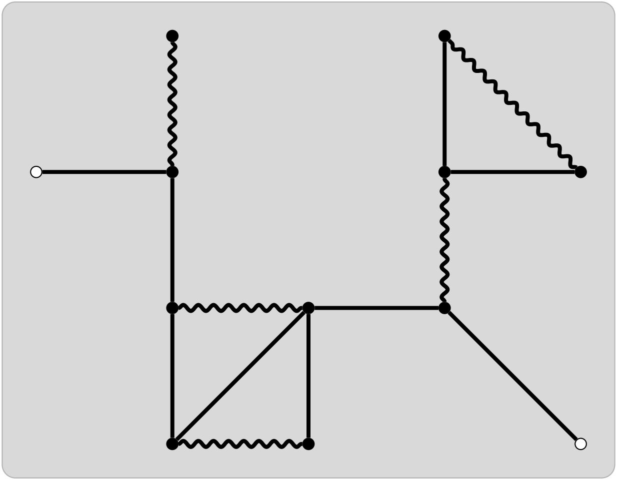

Augmentation $\Rightarrow$ Bigger matching.

Berge and Edmonds' results

Maximum matching = Biggest matching.
$\mu(G)$ = Cardinality of a maximum matching of G.

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Theorem [Berge, 1957]

Maximum matching $\Leftrightarrow$ No augmenting path.

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Finding augmenting paths?

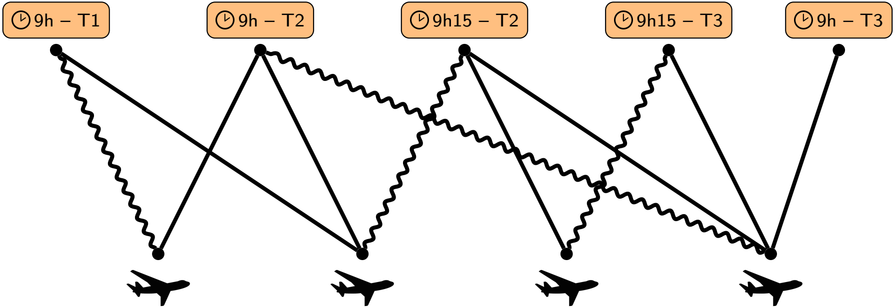
Theorem [Edmonds' Blossom Algorithm, 1965]

Detection in polynomial time.

Hence, $\mu(G)$ can be determined in poly-time.

Today's motivation (let's pretend ☺)

Plane → Suitable landing slot times/tracks (edges) + Scheduled one (matching).



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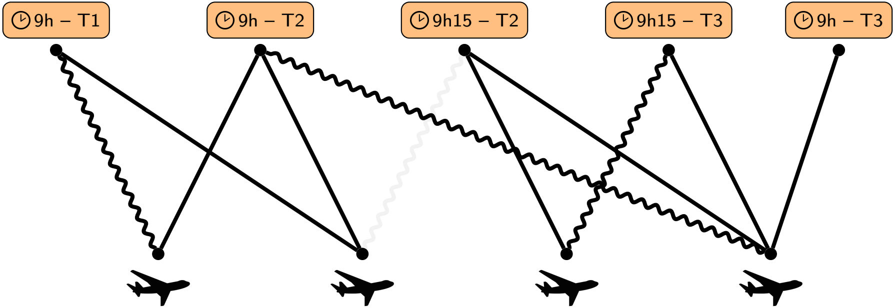
Issue: For some reason, 2nd plane cannot land on Track 2 at 9h15 any more...



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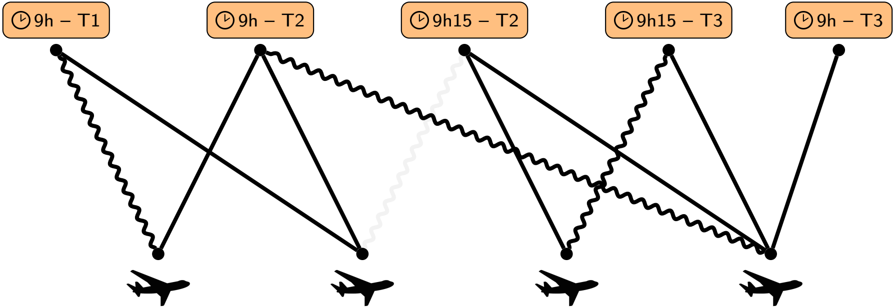
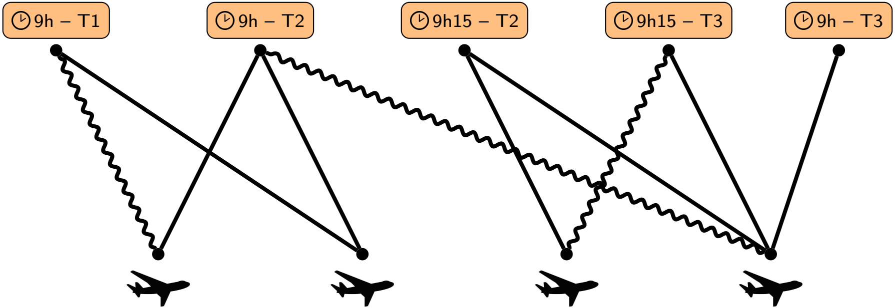


What should we do??

Today's motivation

Re-scheduling a lot is not acceptable! $\Rightarrow$ Cannot start over from scratch.
$\Rightarrow$ Modify the matching "locally", via an augmentation.



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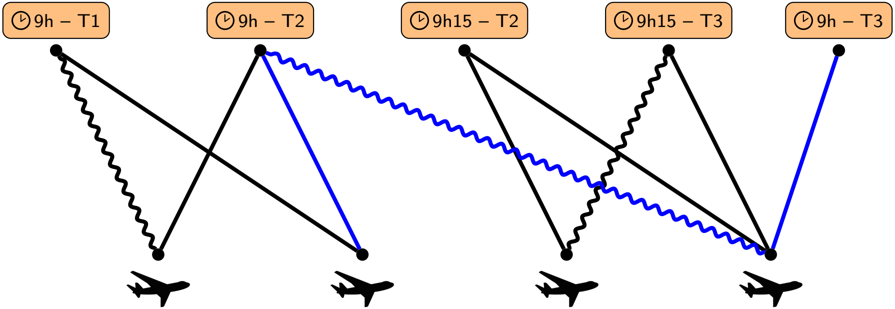

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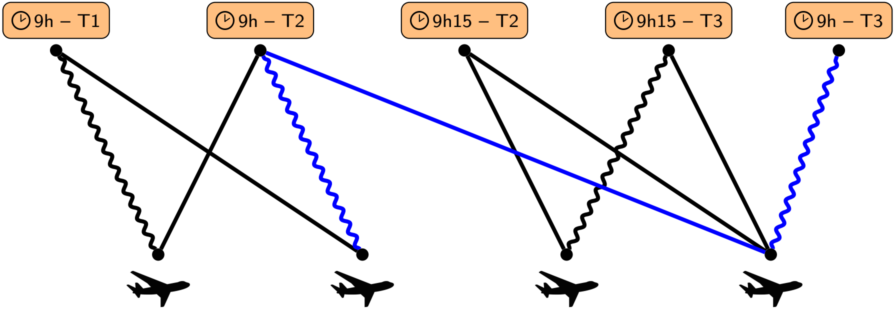

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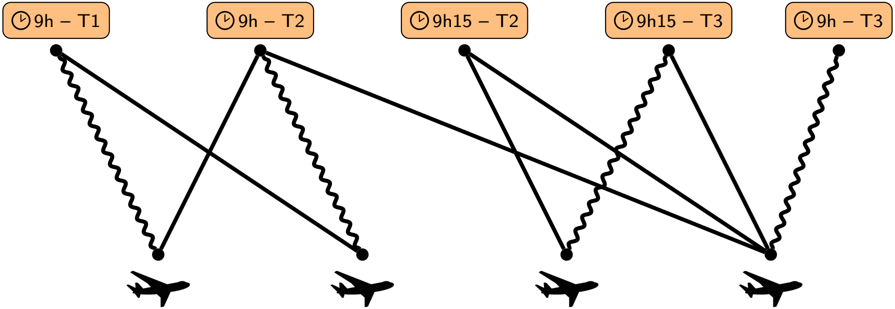


General question

Question

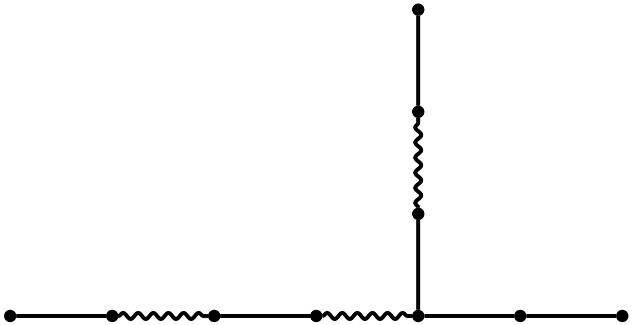
For odd $k \geq 1$, attain a largest matching via $(\leq k)$-augmentations?

$\mu_{\leq k}(G, M)$: Its cardinality for G equipped with M.

Note: $\mu_{\leq 1}(G, \emptyset) = \mu(G)$.

Note: order matters ☹

$k = 5$. First attempt.



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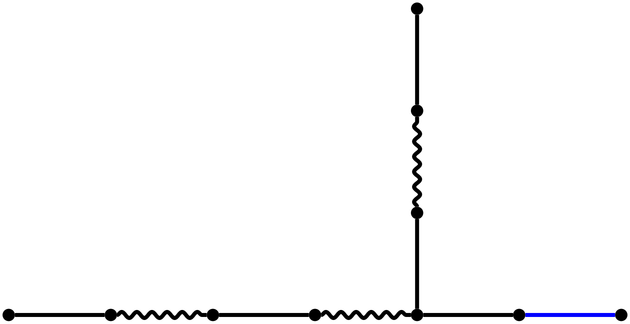

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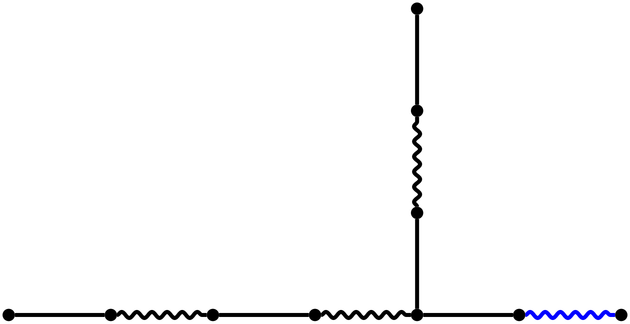

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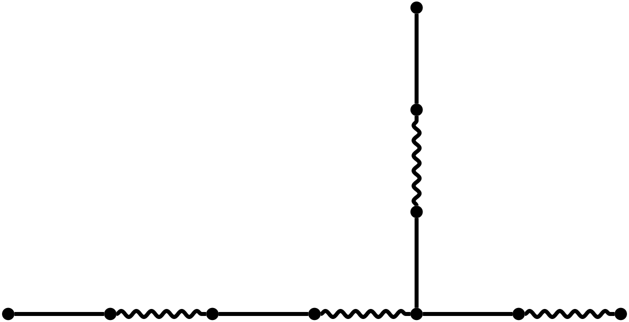

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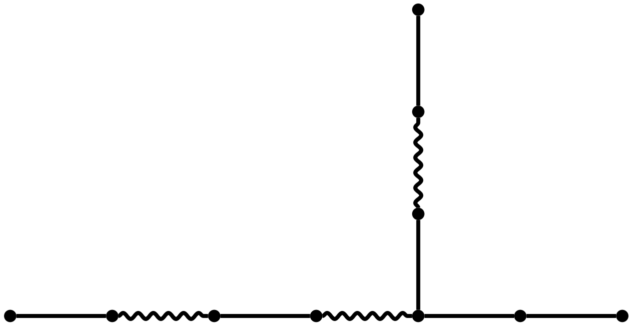

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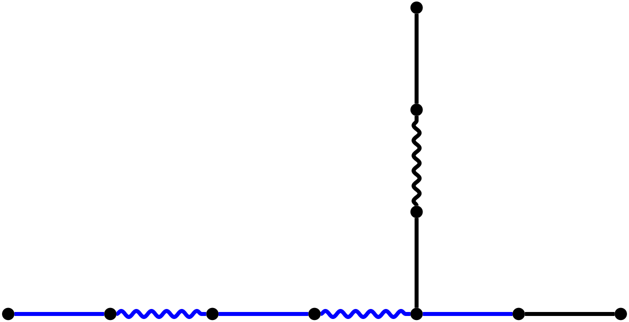

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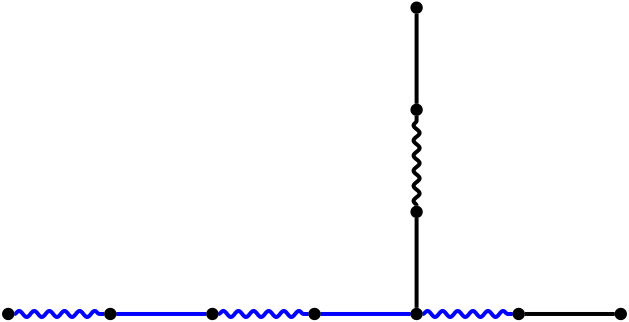

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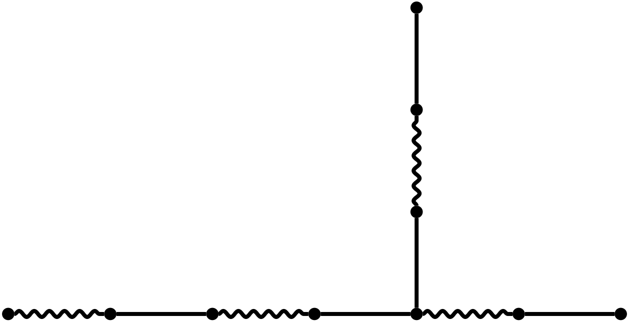

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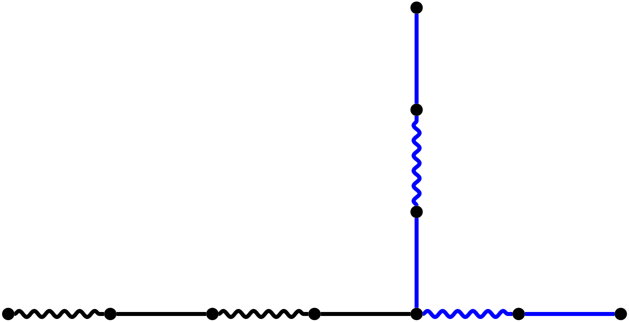

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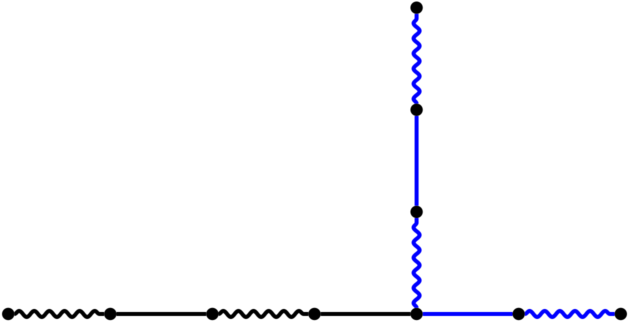

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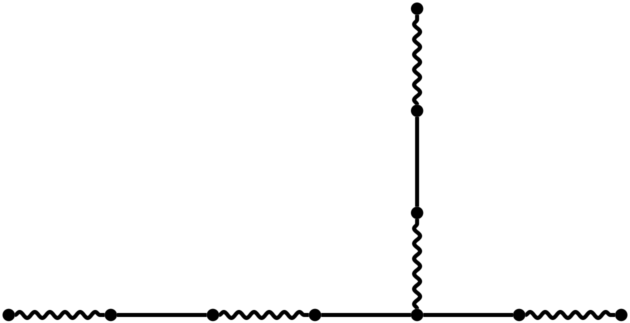

First dichotomy

($\leq k$)-Matching Problem – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{\leq k}(G, M)$?

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For fixed k's, a dichotomy:

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is

- in P for $k = 1, 3$;
- NP-hard for every odd $k \geq 5$.

Latter statement true for planar bipartite graphs with $\Delta \leq 3$ and arb. large girth.

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
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- $(\leq k)$-MP is in P for caterpillars, subdivided stars, "sparse trees", etc.
- A modified version is NP-complete for trees.

Positive results

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

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- In subdivided stars?
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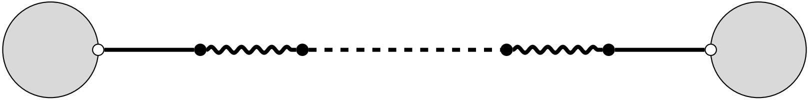
  $\Rightarrow$ Can root-augmentations be avoided?
- Trees where the *b*-vertices are sufficiently far apart?

Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.



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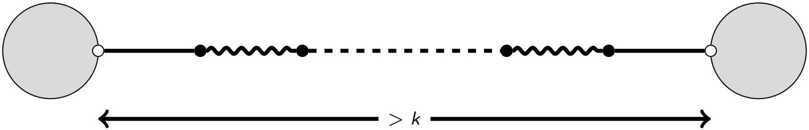

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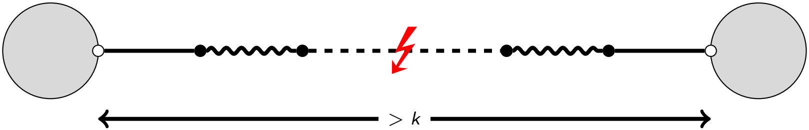

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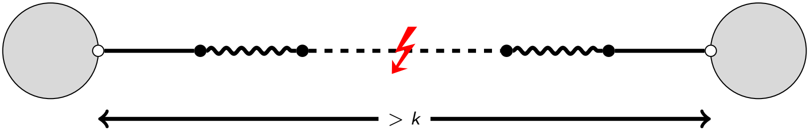

$\Rightarrow$ Decompose the problem into two sub-problems.

In a path $\Rightarrow$ Assume exposed vertices have one on the left/right at distance $\leq k$.

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2nd key idea: We can augment paths joining "consecutive" exposed vertices only.

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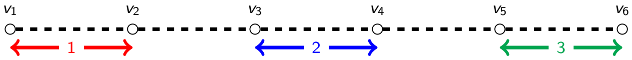

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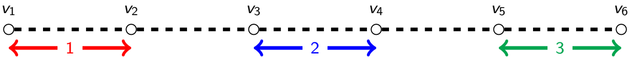


yields the same matching.

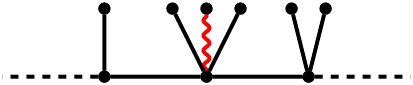
$\Rightarrow$ In a path, just go from left to right, and augment paths when possible. ■

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.



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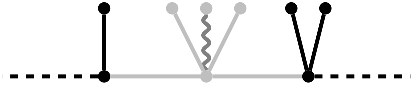

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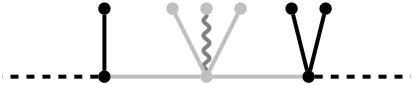

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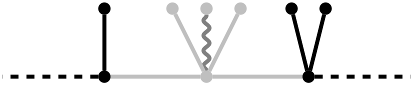

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Focus on caterpillars with $\Delta = 3$ ($\sim$ paths).

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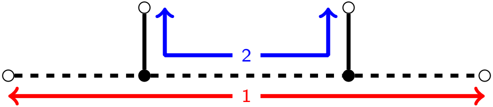

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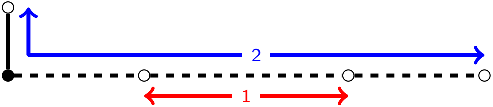

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$\Rightarrow$ Just as for paths, go from left to right (for a specific ordering), and match. ■

Subdivided stars

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Enhancement: Cope with root-augmentations.

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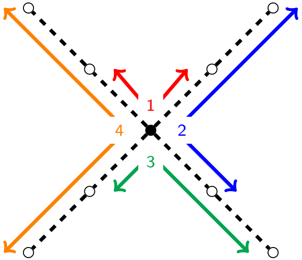

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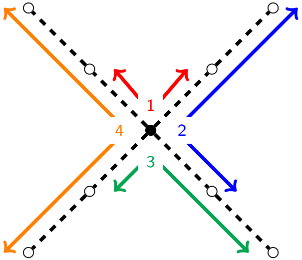

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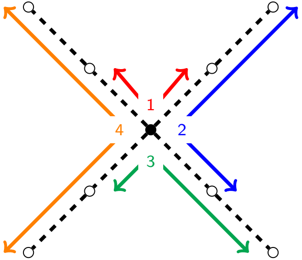

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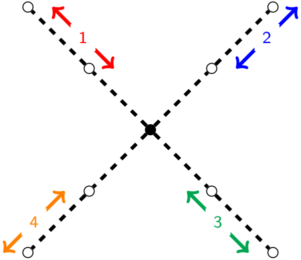

(because 1, 2, 3 and 4 are augmenting $(\leq k)$-paths.)

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for subdivided stars.

So why performing root-augmentations?

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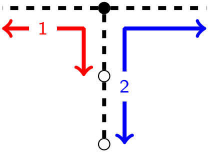

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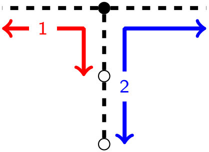

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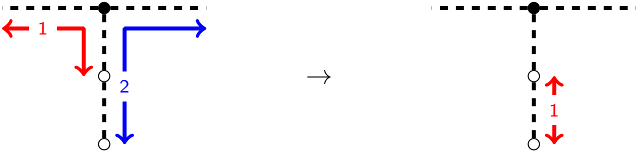

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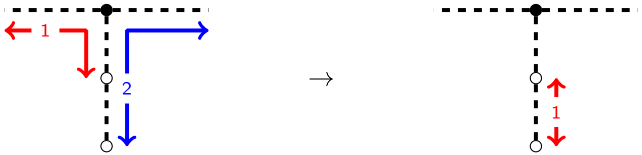

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⇒ Root-augmentation → Alters the parity of the two end-branches only.

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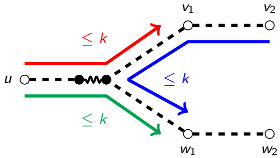

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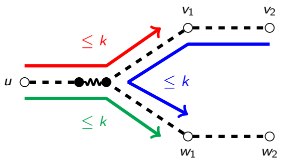

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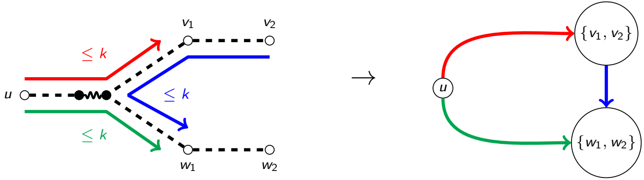

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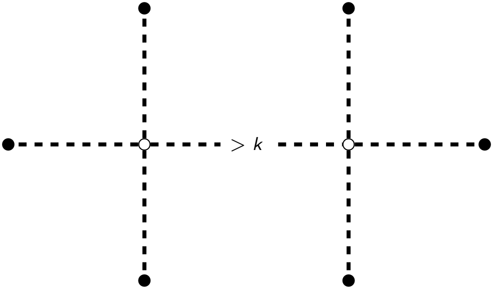
To summarize:

1. If necessary, do an augmentation involving the root.
2. If possible, join two odd branches via root-augmentations.
3. Finally, match the remaining exposed vertices along the branches.

$\Rightarrow$ Polynomial-time algorithm. ∎

Going to sparse trees

k-**sparse tree:** Vertices with degree ≥ 3 are at distance $> k$.

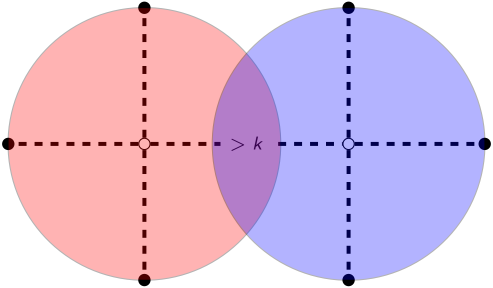


$(\leq k)$-MP for k-sparse trees

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for k-sparse trees.

Idea: Consider subdivided stars, and build a solution from bottom to top. ∎



Negative results

Original intention

NP-hardness proof: Need some forcing mechanisms.

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($= k$)-MATCHING PROBLEM – ($= k$)-MP
Input: A graph G, and a matching M of G.
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Good news: Some properties of $(\leq k)$-MP derive to $(= k)$-MP:

- NP-hardness for odd $k \geq 5$;
- all polynomial-time algorithms for classes of trees.

$(= k)$-MP in trees for non-fixed k

Modified version:

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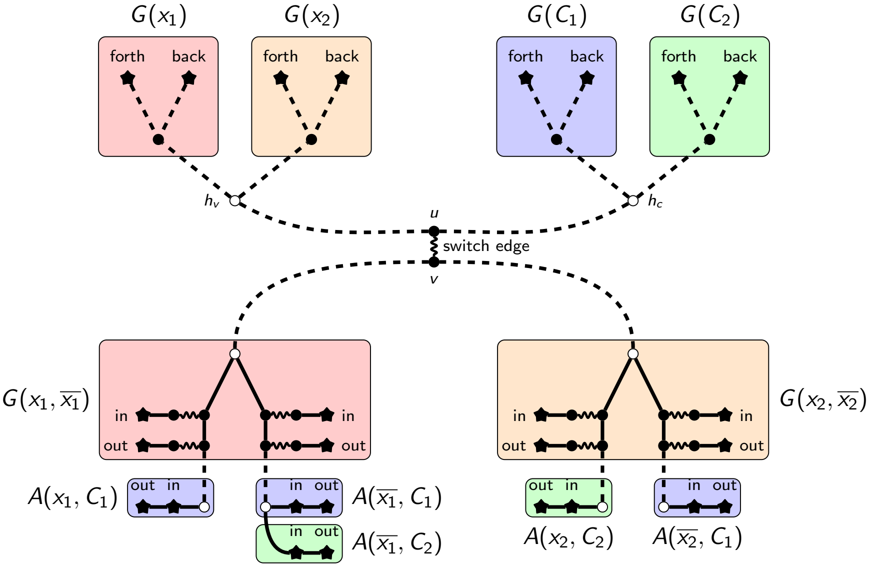
Negative result for trees:

Theorem [B., Garnero, Nisse, 2017+]

$(=)$-MP is NP-hard for trees.

Proof (sketch): Reduction from 3-SAT.

$(=)$-MP in trees



G(x1)
G(x2)
G(C1)
G(C2)
forth
back
hv
hc
u
switch edge
v
G(x1, x̄1)
G(x2, x̄2)
in
out
A(x1, C1)
A(x̄1, C1)
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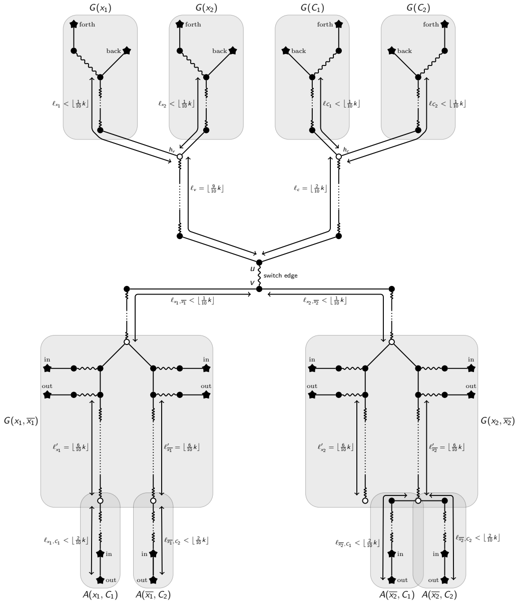
$\Rightarrow$ Needed k depends on #clauses and #variables. ■

After a few months suffering ☺☹ ...



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Thank you for your attention!