Augmenting matchings in trees, via bounded-length augmentations

Julien Bensmail, Valentin Garnero, Nicolas Nisse

Université Nice-Sophia-Antipolis, France

STINT Meeting

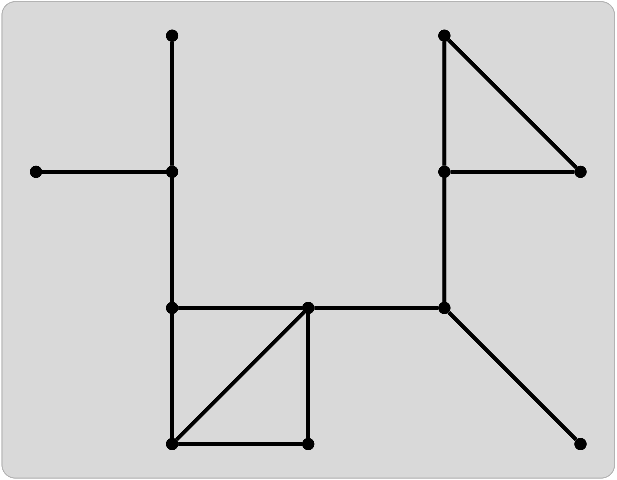
July 5th, 2017

Introduction

Cast

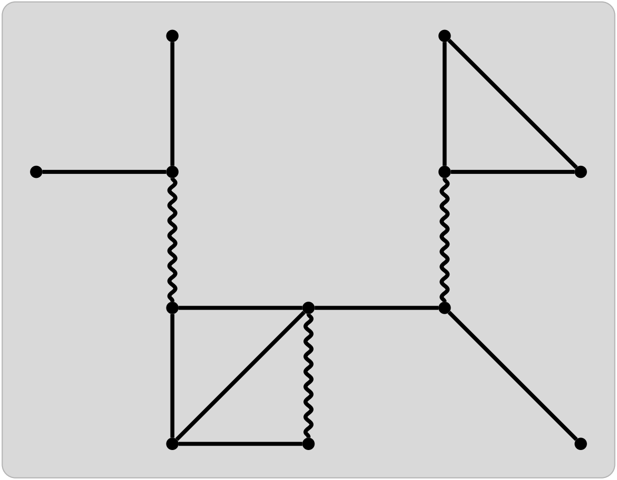
Cast

Graph



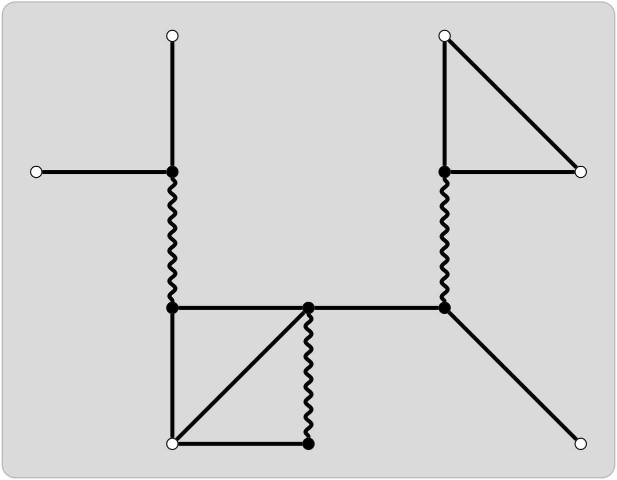
Cast

Graph, Matching.



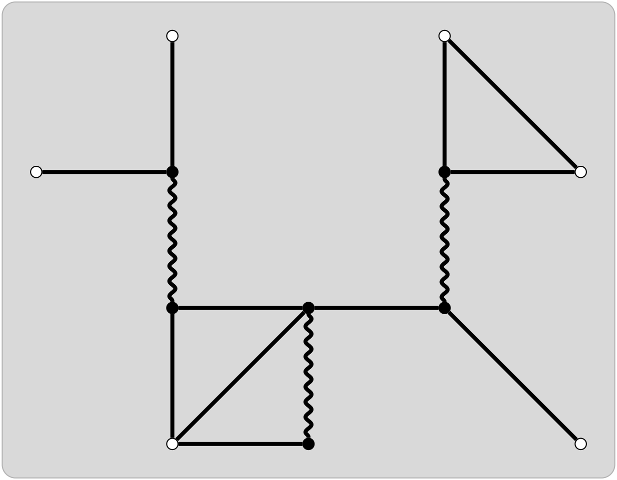
Cast

Graph, Matching.
Exposed vertex, Covered vertex.



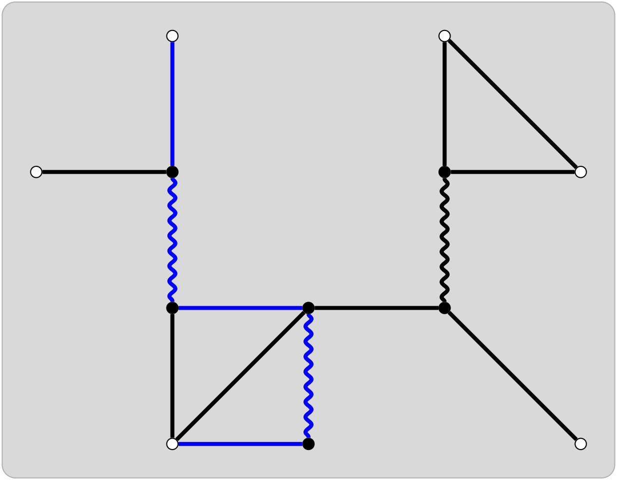
Augmenting a matching

Augmenting path, Augmentation.



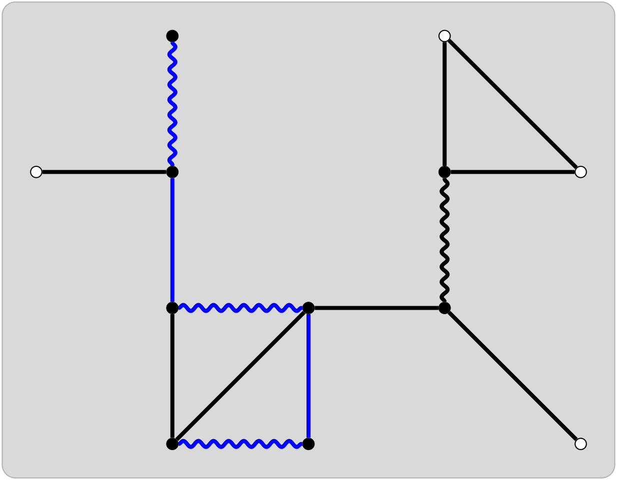
Augmenting a matching

Augmenting path, Augmentation.



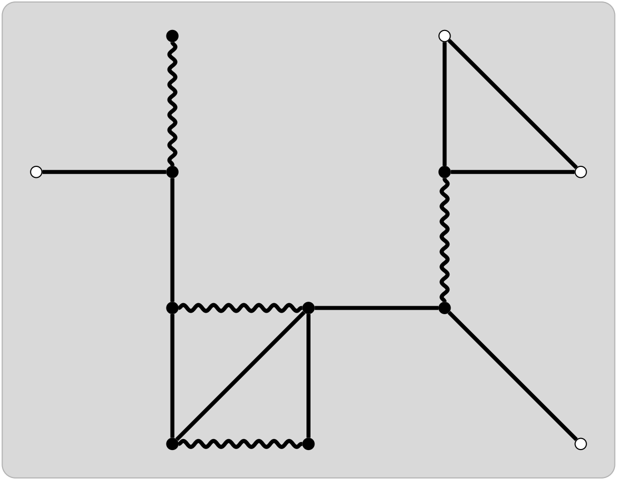
Augmenting a matching

Augmenting path, Augmentation.



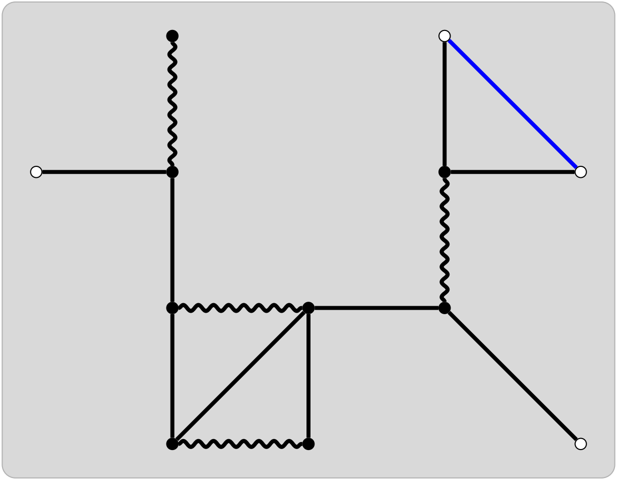
Augmenting a matching

Augmenting path, Augmentation.



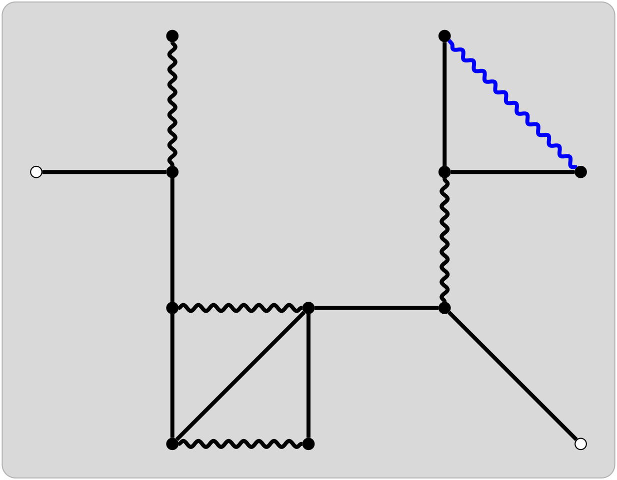
Augmenting a matching

Augmenting path, Augmentation.



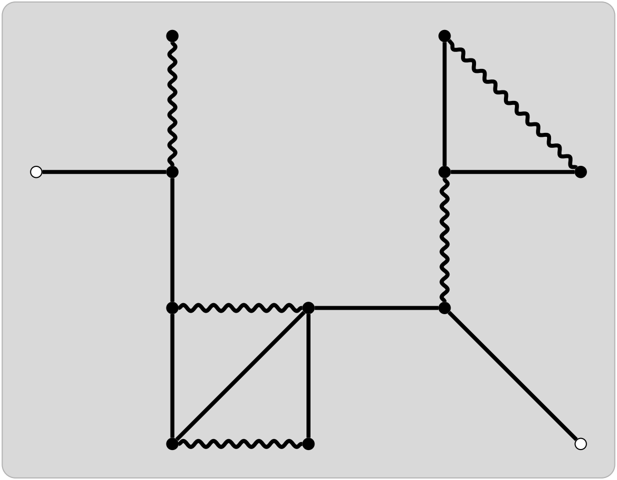
Augmenting a matching

Augmenting path, Augmentation.



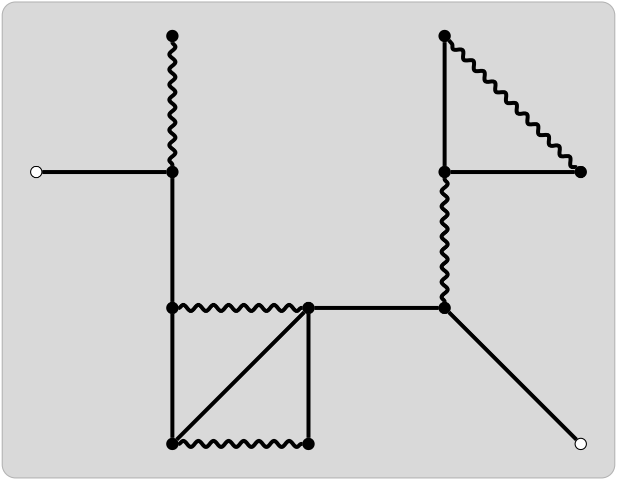
Augmenting a matching

Augmenting path, Augmentation.



Augmenting a matching

Augmenting path, Augmentation.



Augmentation ⇒ Bigger matching.

Berge and Edmonds' results

Maximum matching = Biggest matching.
$\mu(G)$ = Cardinality of a maximum matching of G.

Berge and Edmonds' results

Maximum matching = Biggest matching.
$\mu(G)$ = Cardinality of a maximum matching of G.

Theorem [Berge, 1957]

Maximum matching $\Leftrightarrow$ No augmenting path.

Berge and Edmonds' results

Maximum matching $=$ Biggest matching.
$\mu(G)$ = Cardinality of a maximum matching of G.

Theorem [Berge, 1957]

Maximum matching $\Leftrightarrow$ No augmenting path.

Finding augmenting paths?

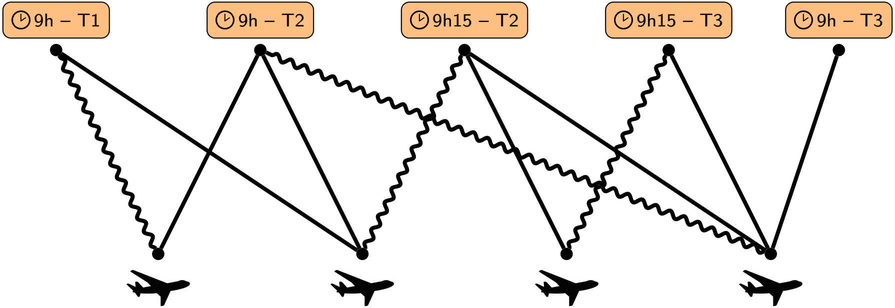
Theorem [Edmonds' Blossom Algorithm, 1965]

Detection in polynomial time.

Hence, $\mu(G)$ can be determined in poly-time.

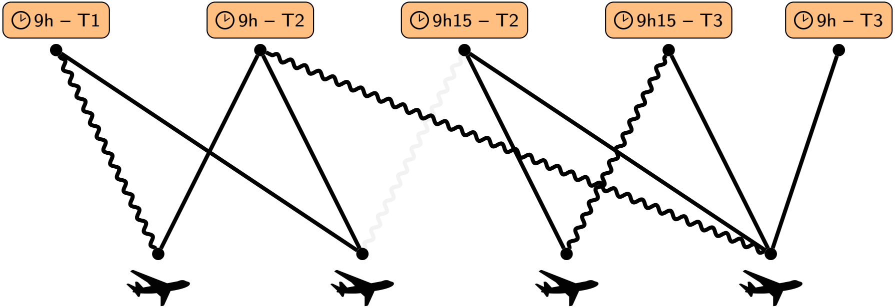
Today's motivation (let's pretend ☺)

Plane → Suitable landing slot times/tracks (edges) + Scheduled one (matching).



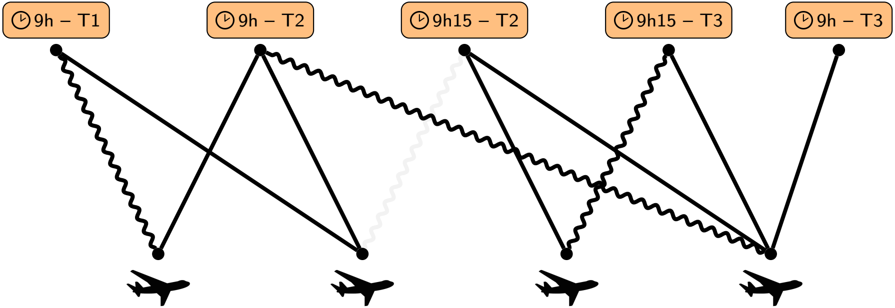
Today's motivation

Issue: For some reason, 2nd plane cannot land on Track 2 at 9h15 any more...



Today's motivation

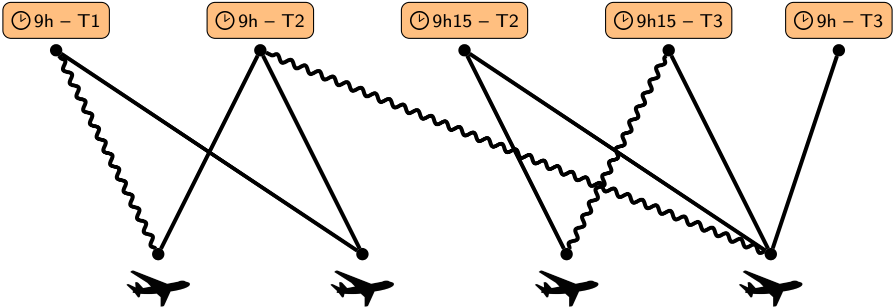
Issue: For some reason, 2nd plane cannot land on Track 2 at 9h15 any more...



What should we do??

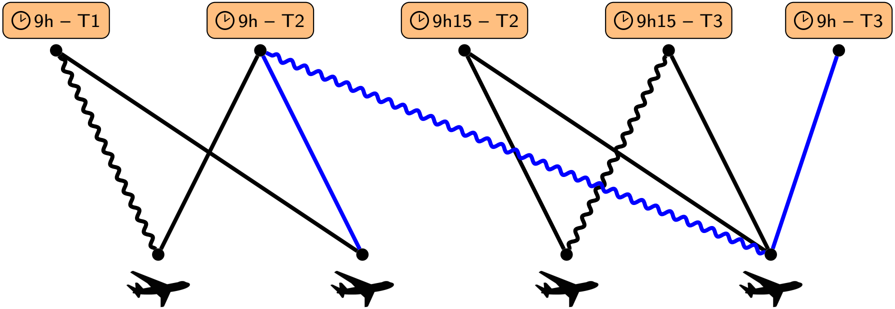
Today's motivation

Re-scheduling a lot is not acceptable! $\Rightarrow$ Cannot start over from scratch.
$\Rightarrow$ Modify the matching "locally", via an augmentation.



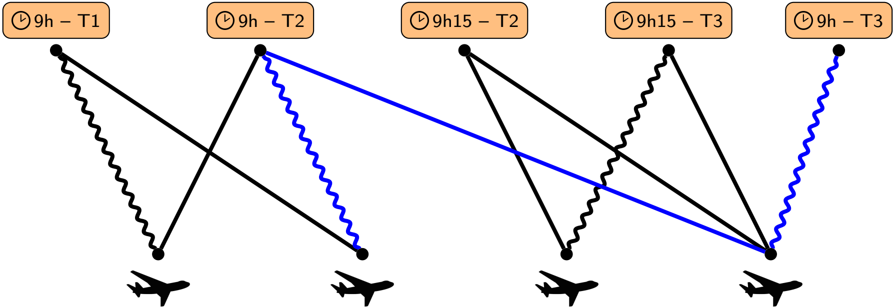
Today's motivation

Re-scheduling a lot is not acceptable! ⇒ Cannot start over from scratch.
⇒ Modify the matching "locally", via an augmentation.



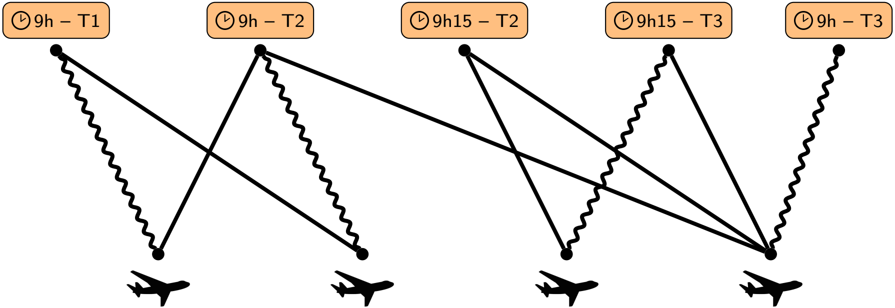
Today's motivation

Re-scheduling a lot is not acceptable! ⇒ Cannot start over from scratch.
⇒ Modify the matching "locally", via an augmentation.



Today's motivation

Re-scheduling a lot is not acceptable! ⇒ Cannot start over from scratch.
⇒ Modify the matching "locally", via an augmentation.



General question

Question

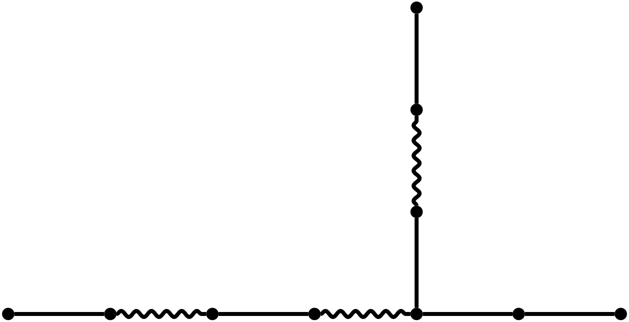
For odd $k \geq 1$, attain a largest matching via $(\leq k)$-augmentations?

$\mu_{\leq k}(G, M)$: Its cardinality for G equipped with M.

Note: $\mu_{\leq 1}(G, \emptyset) = \mu(G)$.

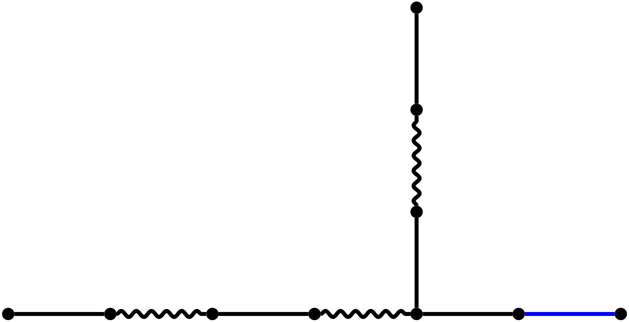
Note: order matters ☹

$k = 5$. First attempt.



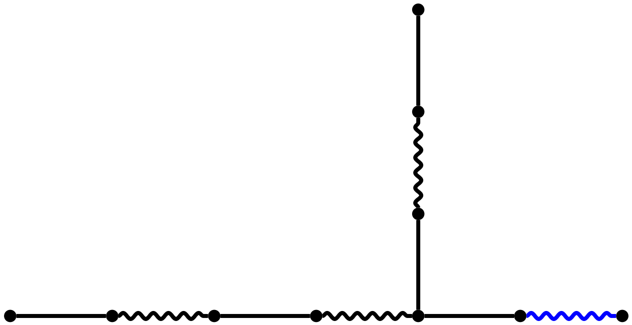
Note: order matters ☹

$k = 5$. First attempt.



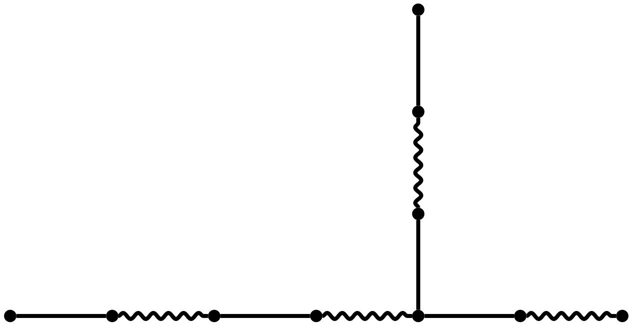
Note: order matters ☹

$k = 5$. First attempt.



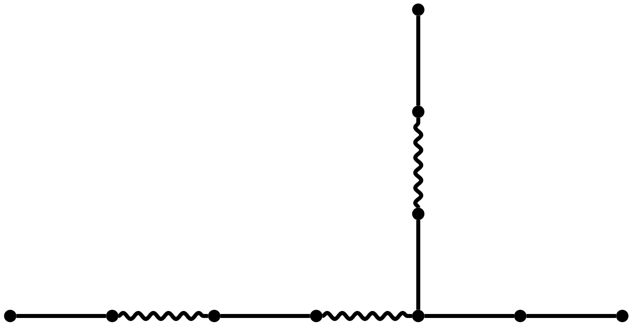
Note: order matters ☹

$k = 5$. First attempt.



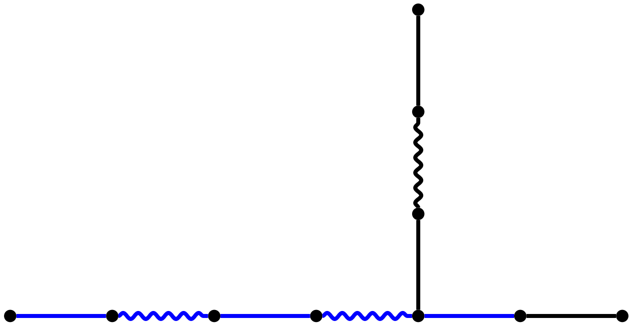
Note: order matters ☹

$k = 5$. Second attempt.



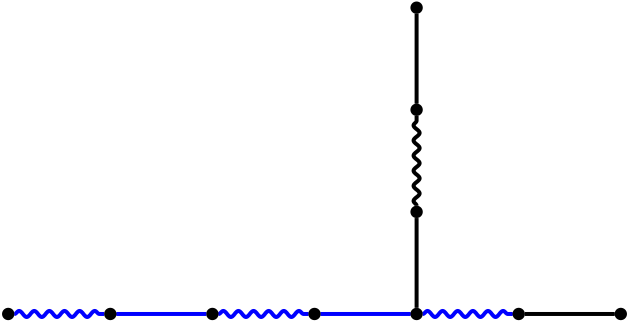
Note: order matters ☹

$k = 5$. Second attempt.



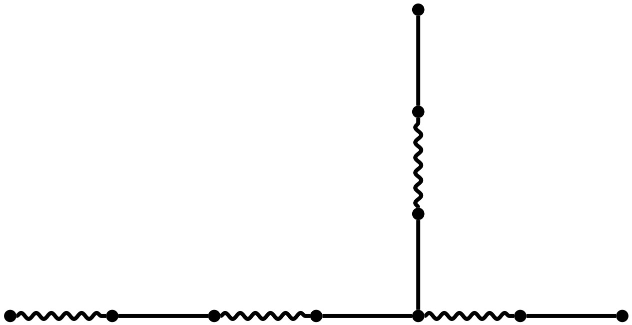
Note: order matters ☹

$k = 5$. Second attempt.



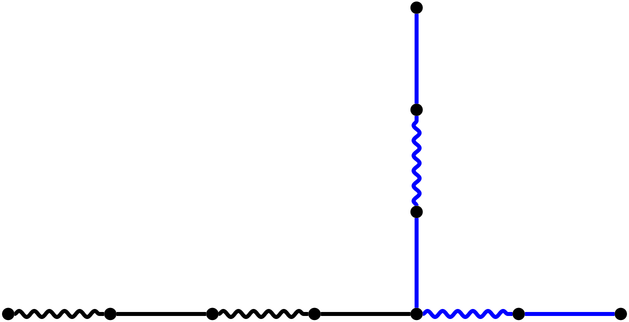
Note: order matters ☹

$k = 5$. Second attempt.



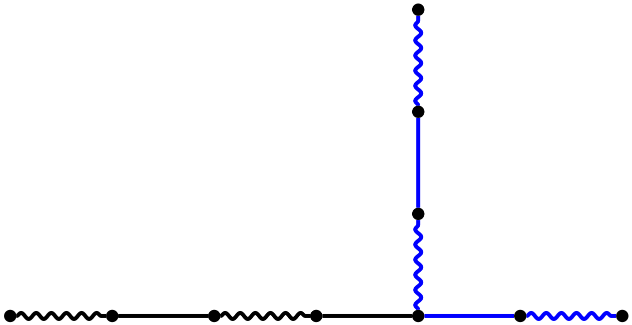
Note: order matters ☹

$k = 5$. Second attempt.



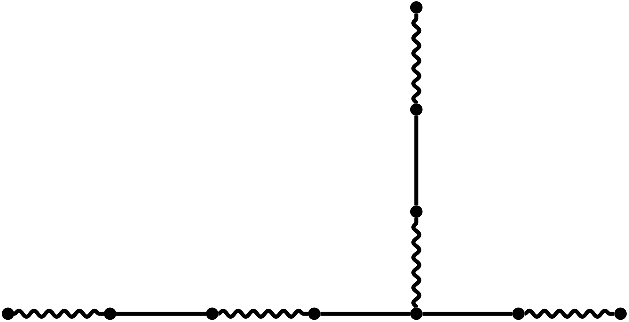
Note: order matters ☹

$k = 5$. Second attempt.



Note: order matters ☹

$k = 5$. Second attempt.



First dichotomy

($\leq k$)-MATCHING PROBLEM – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{\leq k}(G, M)$?

First dichotomy

($\leq k$)-Matching Problem – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{\leq k}(G, M)$?

Reminder: ($\leq \infty$)-MP is in P, by Berge and Edmonds' results.

First dichotomy

($\leq k$)-Matching Problem – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{\leq k}(G, M)$?

Reminder: ($\leq \infty$)-MP is in P, by Berge and Edmonds' results.

For fixed k's, a dichotomy:

Theorem [Nisse, Salch, Weber, 2015+]

($\leq k$)-MP is

- in P for $k = 1, 3$;
- NP-hard for every odd $k \geq 5$.

Latter statement true for planar bipartite graphs with $\Delta \leq 3$ and arb. large girth.

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
- For odd $k \geq 5$, NP-hard for graphs close to trees.

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
- For odd $k \geq 5$, NP-hard for graphs close to trees.

Complexity of $(\leq k)$-MP for trees?

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
- For odd $k \geq 5$, NP-hard for graphs close to trees.

Complexity of $(\leq k)$-MP for trees?

Today's talk:

- $(\leq k)$-MP is in P for caterpillars, subdivided stars, "sparse trees", etc.

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
- For odd $k \geq 5$, NP-hard for graphs close to trees.

Complexity of $(\leq k)$-MP for trees?

Today's talk:

- $(\leq k)$-MP is in P for caterpillars, subdivided stars, "sparse trees", etc.
- A modified version is NP-complete for trees.

Positive results

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

- In paths, augmenting path overlaps can be avoided.

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

- In paths, augmenting path overlaps can be avoided.
  $\Rightarrow$ Go from "left to right" and "match" consecutive exposed vertices.

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

- In paths, augmenting path overlaps can be avoided.
  $\Rightarrow$ Go from "left to right" and "match" consecutive exposed vertices.
- Can it be done in caterpillars?

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

- In paths, augmenting path overlaps can be avoided.
  $\Rightarrow$ Go from "left to right" and "match" consecutive exposed vertices.
- Can it be done in caterpillars?
- In subdivided stars?

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

- In paths, augmenting path overlaps can be avoided.
  $\Rightarrow$ Go from "left to right" and "match" consecutive exposed vertices.
- Can it be done in caterpillars?
- In subdivided stars?
  $\Rightarrow$ Augmentations along branches $\Leftrightarrow$ Path case.
  $\Rightarrow$ Can root-augmentations be avoided?

Intuition (= spoilers)

One key idea: Prove that $\exists$ a particular way to reach a max. matching.

Upcoming ideas:

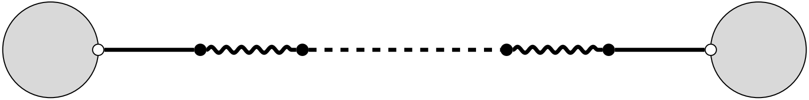
- In paths, augmenting path overlaps can be avoided.
  $\Rightarrow$ Go from "left to right" and "match" consecutive exposed vertices.
- Can it be done in caterpillars?
- In subdivided stars?
  $\Rightarrow$ Augmentations along branches $\Leftrightarrow$ Path case.
  $\Rightarrow$ Can root-augmentations be avoided?
- Trees where the *b*-vertices are sufficiently far apart?

Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.

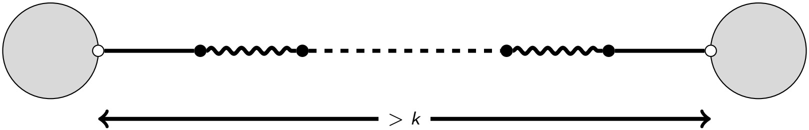


Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.

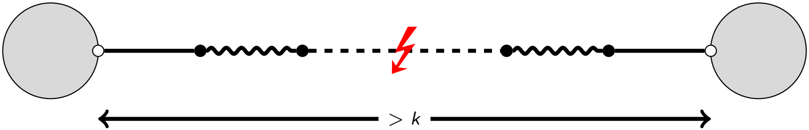


Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.

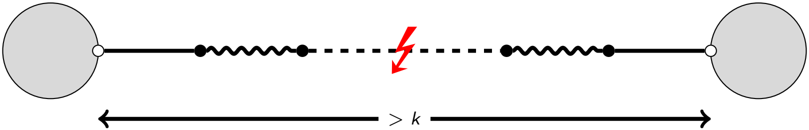


Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

($\leq k$)-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.



$\Rightarrow$ Decompose the problem into two sub-problems.

In a path $\Rightarrow$ Assume exposed vertices have one on the left/right at distance $\leq k$.

Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

2nd key idea: We can augment paths joining "consecutive" exposed vertices only.

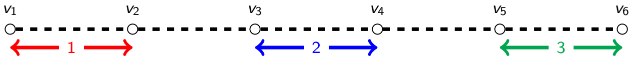
Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

2nd key idea: We can augment paths joining "consecutive" exposed vertices only.

3 $\Rightarrow$ The paths $v_1 \ldots v_2$, $v_3 \ldots v_4$ and $v_5 \ldots v_6$ have length $\leq k$ and alternate. So



yields the same matching.

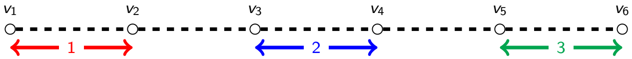
Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

2nd key idea: We can augment paths joining "consecutive" exposed vertices only.

3 $\Rightarrow$ The paths $v_1 \ldots v_2$, $v_3 \ldots v_4$ and $v_5 \ldots v_6$ have length $\leq k$ and alternate. So



yields the same matching.

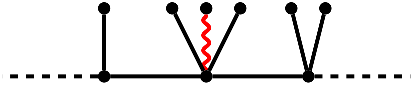
$\Rightarrow$ In a path, just go from left to right, and augment paths when possible. ■

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.

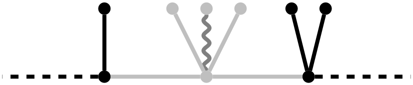


Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.

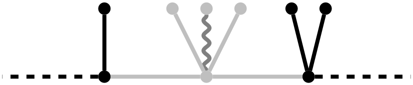


Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.



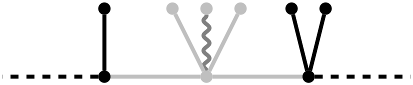
$\Rightarrow$ Being adjacent to two leaves is useless.

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.



$\Rightarrow$ Being adjacent to two leaves is useless.

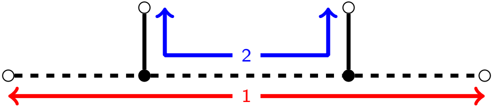
Focus on caterpillars with $\Delta = 3$ ($\sim$ paths).

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Again, augmenting paths can be “disentangled”:



Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Again, augmenting paths can be "disentangled":

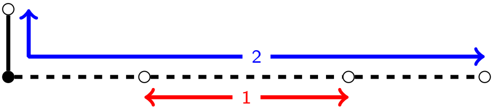
Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for caterpillars.

Again, augmenting paths can be “disentangled”:

and



Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Again, augmenting paths can be “disentangled”:

and

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for caterpillars.

Again, augmenting paths can be "disentangled":

and

$\Rightarrow$ Just as for paths, go from left to right (for a specific ordering), and match. ■

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Enhancement: Cope with root-augmentations.

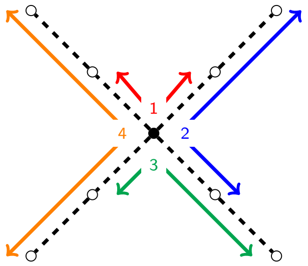
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Enhancement: Cope with root-augmentations.

Key fact: "Looping" root-augmentations can be avoided:



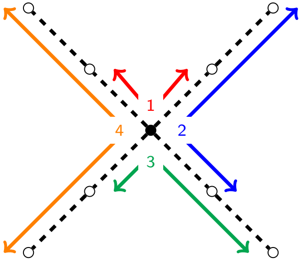
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Enhancement: Cope with root-augmentations.

Key fact: "Looping" root-augmentations can be avoided:



equivalent to

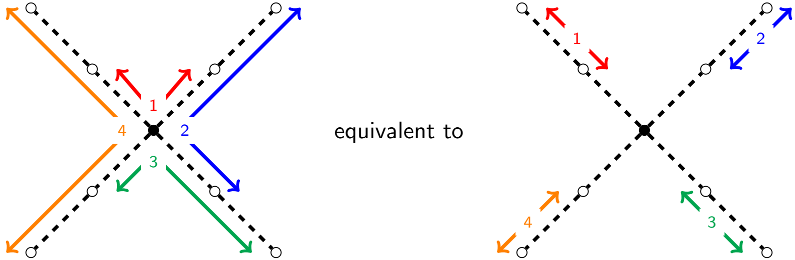
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Enhancement: Cope with root-augmentations.

Key fact: "Looping" root-augmentations can be avoided:



(because 1, 2, 3 and 4 are augmenting $(\leq k)$-paths.)

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

So why performing root-augmentations?

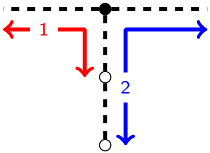
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

So why performing root-augmentations?

From the p.o.v. of an inner-branch, an equivalent augmentation can be performed



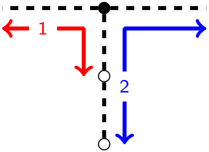
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

So why performing root-augmentations?

From the p.o.v. of an inner-branch, an equivalent augmentation can be performed



$\rightarrow$

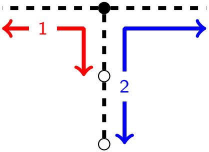
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

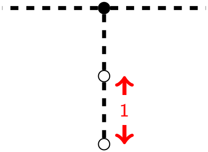
$(\leq k)$-MP is in P for subdivided stars.

So why performing root-augmentations?

From the p.o.v. of an inner-branch, an equivalent augmentation can be performed



$\rightarrow$



... and it retains the **parity** of the number of exposed vertices along that branch.

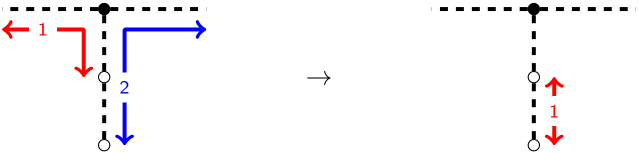
Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

So why performing root-augmentations?

From the p.o.v. of an inner-branch, an equivalent augmentation can be performed



... and it retains the **parity** of the number of exposed vertices along that branch.

$\Rightarrow$ Root-augmentation $\rightarrow$ Alters the parity of the two end-branches only.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

So, we can reach a maximum matching by essentially:

1. Performing root-augmentations to match vertices from $\neq$ **odd** branches;

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

So, we can reach a maximum matching by essentially:

1. Performing root-augmentations to match vertices from $\neq$ **odd** branches;
2. Then finishing off along the branches.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

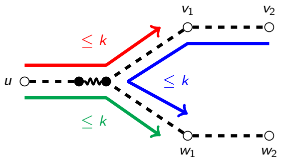
$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

So, we can reach a maximum matching by essentially:

1. Performing root-augmentations to match vertices from $\neq$ **odd** branches;
2. Then finishing off along the branches.

To check if 1. doable, run a BFS in an auxiliary “reachability digraph”:



Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

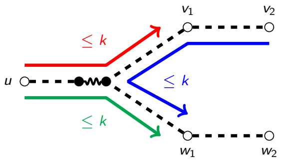
($\leq k$)-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

So, we can reach a maximum matching by essentially:

1. Performing root-augmentations to match vertices from $\neq$ **odd** branches;
2. Then finishing off along the branches.

To check if 1. doable, run a BFS in an auxiliary "reachability digraph":



$\rightarrow$

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

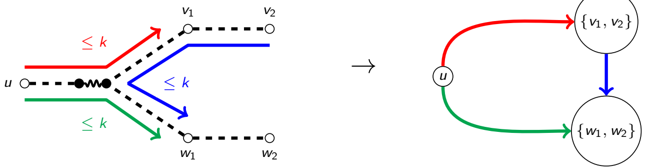
$(\leq k)$-MP is in P for subdivided stars.

Remind that for a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations.
$\Rightarrow$ No point starting/ending with an even branch.

So, we can reach a maximum matching by essentially:

1. Performing root-augmentations to match vertices from $\neq$ **odd** branches;
2. Then finishing off along the branches.

To check if 1. doable, run a BFS in an auxiliary "reachability digraph":



Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for subdivided stars.

To summarize:

1. If necessary, do an augmentation involving the root.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

To summarize:

1. If necessary, do an augmentation involving the root.
2. If possible, join two odd branches via root-augmentations.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

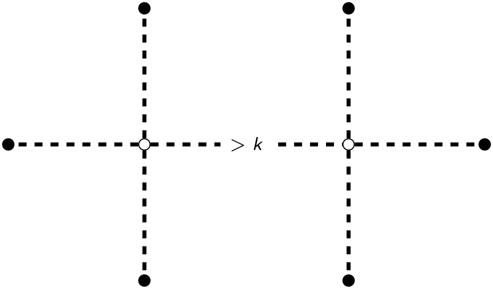
To summarize:

1. If necessary, do an augmentation involving the root.
2. If possible, join two odd branches via root-augmentations.
3. Finally, match the remaining exposed vertices along the branches.

$\Rightarrow$ Polynomial-time algorithm. ∎

Going to sparse trees

k-**sparse tree:** Vertices with degree ≥ 3 are at distance $> k$.

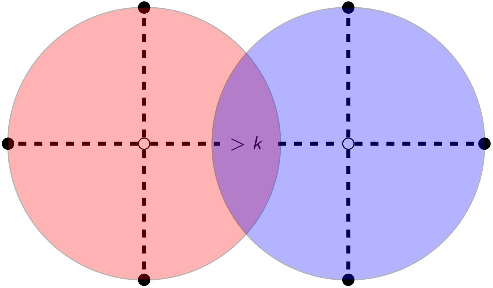


$(\leq k)$-MP for k-sparse trees

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for k-sparse trees.

Idea: Consider subdivided stars, and build a solution from bottom to top. ■



Negative results

Original intention

NP-hardness proof: Need some forcing mechanisms.

For $(\leq k)$-MP in trees, sounds hard because of the "$\leq k$" requirement ☹.

Original intention

NP-hardness proof: Need some forcing mechanisms.

For ($\leq k$)-MP in trees, sounds hard because of the "$\leq k$" requirement ☹.

$\Rightarrow$ What if we augment k-paths only?

Original intention

NP-hardness proof: Need some forcing mechanisms.

For $(\leq k)$-MP in trees, sounds hard because of the "$\leq k$" requirement ☹.

$\Rightarrow$ What if we augment k-paths only?

$(=k)$-Matching Problem – $(=k)$-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{=k}(G, M)$?

Original intention

NP-hardness proof: Need some forcing mechanisms.

For $(\leq k)$-MP in trees, sounds hard because of the "$\leq k$" requirement ☹.

$\Rightarrow$ What if we augment k-paths only?

$(= k)$-Matching Problem – $(= k)$-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{=k}(G, M)$?

Good news: Some properties of $(\leq k)$-MP derive to $(= k)$-MP:

- NP-hardness for odd $k \geq 5$;
- all polynomial-time algorithms for classes of trees.

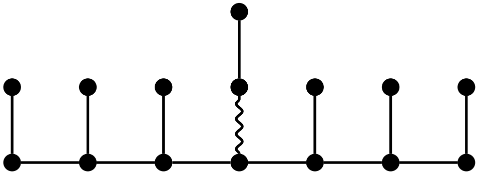
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

(Rough) proof: Reduction from 3-SAT. Just need variable gadget:



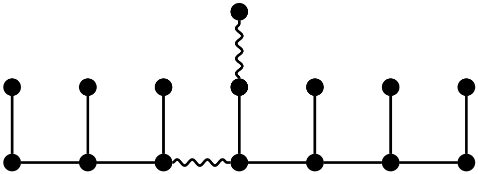
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

(Rough) proof: Reduction from 3-SAT. Just need variable gadget:



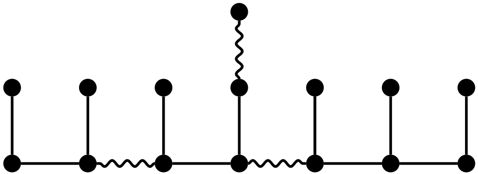
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

(Rough) proof: Reduction from 3-SAT. Just need variable gadget:



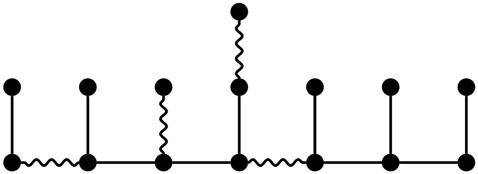
On ($\leq$ 3)-MP and (= 3)-MP

Recall that ($\leq$ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

(= 3)-MP is NP-hard.

(Rough) proof: Reduction from $3\text{-}\mathrm{SAT}$. Just need variable gadget:



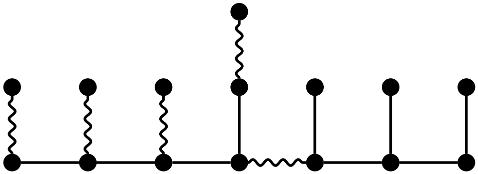
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

(Rough) proof: Reduction from 3-SAT. Just need variable gadget:



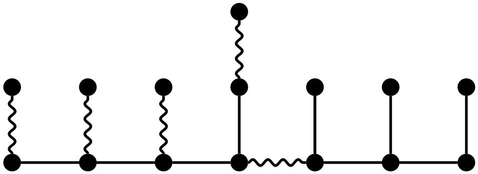
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

(Rough) proof: Reduction from 3-SAT. Just need variable gadget:



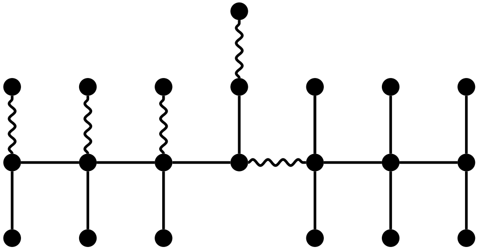
Longest sequence = "Push" the matching to the spikes of a single side.

On (≤ 3)-MP and ($= 3$)-MP

Theorem [B., Garnero, Nisse, 2017+]

($= 3$)-MP is NP-hard.

Attach a leaf to the base of every spike. Previous remark still applies.



On (≤ 3)-MP and $(= 3)$-MP

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Add a variable gadget G_i for each x_i. Pushing left=*true*. Pushing right=*false*.

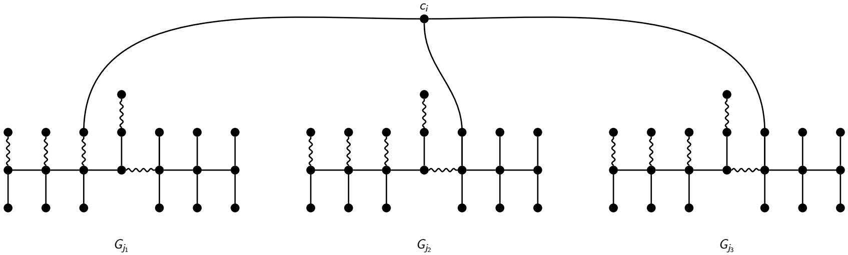
On (≤ 3)-MP and $(= 3)$-MP

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Add a variable gadget G_i for each x_i. Pushing left=*true*. Pushing right=*false*.

Next add a clause vertex c_i for every clause C_i, and, for every distinct literal ℓ_j it contains, join c_i and one non-used spike of G_i (left if positive, right otherwise).



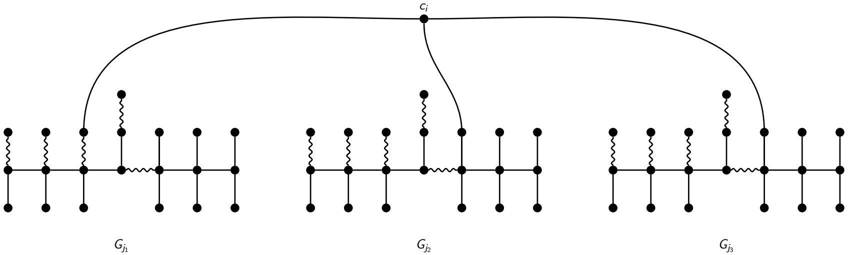
On (≤ 3)-MP and $(= 3)$-MP

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Add a variable gadget G_i for each x_i. Pushing left=*true*. Pushing right=*false*.

Next add a clause vertex c_i for every clause C_i, and, for every distinct literal ℓ_j it contains, join c_i and one non-used spike of G_i (left if positive, right otherwise).



$\Rightarrow$ One additional augmentation covering c_i can be done.

On (≤ 3)-MP and $(= 3)$-MP

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Maximum $\#$ of 3-augmentations is:

1. For every G_i, push the matching to the left (x_i *true*) or to the right (x_i *false*).
2. For every c_i, do an additional augmentation (if made *true* by a literal).

On (≤ 3)-MP and $(= 3)$-MP

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Maximum # of 3-augmentations is:

1. For every G_i, push the matching to the left (x_i *true*) or to the right (x_i *false*).
2. For every c_i, do an additional augmentation (if made *true* by a literal).

$\Rightarrow$ Maximum $\mu_{=3}$ achievable is

$$(\#\text{variables} \cdot (\#\text{variable spikes} + 1)) + \#\text{clauses},$$

which is attainable iff F is satisfiable. ∎

About the reduction

We have $\Delta \leq 4$ in the reduction.

About the reduction

We have $\Delta \leq 4$ in the reduction. Furthermore:

- If F planar, then the reduced graph is planar.

About the reduction

We have $\Delta \leq 4$ in the reduction. Furthermore:

- If F planar, then the reduced graph is planar.
- All cycles go through c_i's and variables gadgets:

About the reduction

We have $\Delta \leq 4$ in the reduction. Furthermore:

- If F planar, then the reduced graph is planar.
- All cycles go through c_i's and variables gadgets:
  - Conveniently choose the joined spikes $\Rightarrow$ Bipartite.
  - Same $\Rightarrow$ Arbitrarily large girth.

About the reduction

We have $\Delta \leq 4$ in the reduction. Furthermore:

- If F planar, then the reduced graph is planar.
- All cycles go through c_i's and variables gadgets:
  - Conveniently choose the joined spikes $\Rightarrow$ Bipartite.
  - Same $\Rightarrow$ Arbitrarily large girth.

$+$ by slight modifications, we can also guarantee $\Delta \leq 3$.

$(= k)$-MP in trees for non-fixed k

Modified version:

(=)-MATCHING PROBLEM – (=)-MP
Input: A graph G, a matching M of G, and an odd $k \geq 1$.
Question: What is the value of $\mu_{=k}(G, M)$?

$(=k)$-MP in trees for non-fixed k

Modified version:

$(=)$-MATCHING PROBLEM – $(=)$-MP
Input: A graph G, a matching M of G, and an odd $k \geq 1$.
Question: What is the value of $\mu_{=k}(G, M)$?

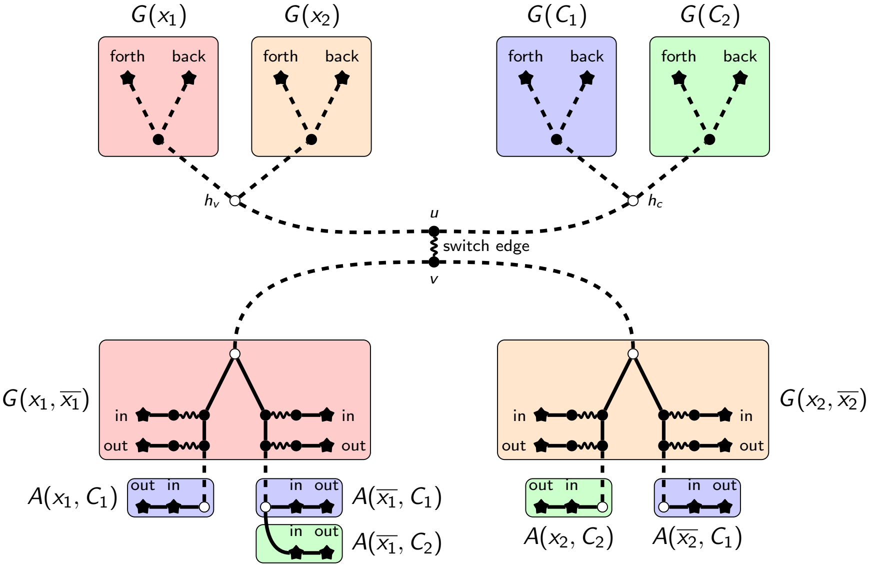
Negative result for trees:

Theorem [B., Garnero, Nisse, 2017+]

$(=)$-MP is NP-hard for trees.

Proof (sketch): Reduction from 3-SAT.

(=)-MP in trees



$G(x_1)$
$G(x_2)$
$G(C_1)$
$G(C_2)$
forth back
forth back
forth back
forth back
h_v
h_c
u
switch edge
v
$G(x_1, \overline{x_1})$
$G(x_2, \overline{x_2})$
in
out
in
out
in
out
in
out
$A(x_1, C_1)$
out in
in out
$A(\overline{x_1}, C_1)$
in out
$A(\overline{x_1}, C_2)$
out in
$A(x_2, C_2)$
in out
$A(\overline{x_2}, C_1)$

$(=)$-MP in trees

Theorem [B., Garnero, Nisse, 2017+]

$(=)$-MP is NP-hard for trees.

Lengths of the dashed paths chosen so that:

- for each x_i, open either the *true* or *false* gate;
- for each C_i, reach only the arrival points.

$(=)$-MP in trees

Theorem [B., Garnero, Nisse, 2017+]

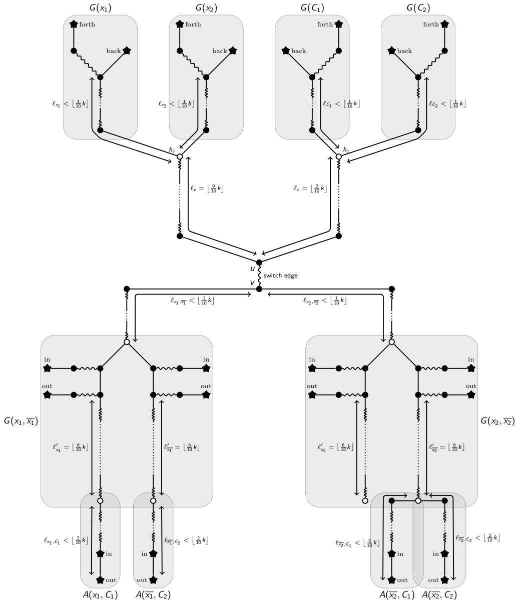
$(=)$-MP is NP-hard for trees.

Lengths of the dashed paths chosen so that:

- for each x_i, open either the *true* or *false* gate;
- for each C_i, reach only the arrival points.

$\Rightarrow$ Needed k depends on #clauses and #variables. ∎

After a few months suffering ☺☹ ...



Conclusion

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?
  - etc.

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?
  - etc.
- Dynamic programming yields algorithms.

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?
  - etc.
- Dynamic programming yields algorithms.
- What about $(= k)$-MP in trees?
- Appealing case: $k = 3$.

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?
  - etc.
- Dynamic programming yields algorithms.
- What about $(= k)$-MP in trees?
- Appealing case: $k = 3$.
- Other classes of graphs?
- e.g. interval graphs, other sparse classes, etc.

Conclusion and perspectives

- Status of $(\leq k)$-MP still unclear for trees.
- What about:
  - trees with $\Delta \leq 3$?
  - subdivided combs?
  - etc.
- Dynamic programming yields algorithms.
- What about $(= k)$-MP in trees?
- Appealing case: $k = 3$.
- Other classes of graphs?
- e.g. interval graphs, other sparse classes, etc.

Thank you for your attention!