Hi Youtube ☺ , we are currently setting things up. We should be able to start soon!

– HDR defence –

A contribution to distinguishing labellings of graphs

Online talk – December 15, 2020

Julien Bensmail

Université Côte d'Azur, France

Jury members:

- **Reviewers:**
  - Mickaël Montassier — PU, Université de Montpellier, France
  - Olivier Togni — PU, Université de Bourgogne, France
  - Xuding Zhu — Prof., National Sun Yat-sen University, Taiwan
- **Examiners:**
  - Cristina Bazgan — PU, Université Paris-Dauphine, France
  - Frédéric Cazals — DR INRIA, INRIA, Sophia Antipolis, France
  - Louis Esperet — CR HDR CNRS, G-SCOP, Grenoble, France
  - Éric Sopena — PU, Université de Bordeaux, France
  - Stéphan Thomassé — PU, ÉNS de Lyon, France

Me in 30 seconds (or surely more ☺)

- Since September 2016, **Maître de conférences** at Université Côte d'Azur
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    w/ C. Thomassen, research group AlgoLoG
  - 2014-2015: **ATER** at LIP/ÉNS de Lyon
    w/ S. Thomassé and N. Trotignon, research group MC2
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    w/ O. Baudon and É. Sopena, research group Graphes et Applications
- **Scientific interests:** graph theory, especially colouring, partitioning, decomposition problems (structural, algorithmic aspects)

Today's talk

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    - The importance of label 3 for the 1-2-3 Conjecture
  - Some mid-term perspectives

A disclaimer before we go

- If you have been through the manuscript...
  - Modified terminology:
    - ~~weight~~, ~~edge-weighting~~ → label, labelling
    - ~~neighbour-sum-distinguishing~~ → proper

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- Results of collaborations led during my successive positions, with *Barme*, Baudon, *Fioravantes*, Hocquard, *Lyngsie*, *Mc Inerney*, *Merker*, Nisse, Przybyło, *Senhaji*, Sopena, Thomassen, Woźniak, etc.

  No offense if your name should be here but is not ☺

From distinguishing labellings to the 1-2-3 Conjecture

(Very general) Motivation

"Encode" a proper vertex-colouring through labels assigned by an (edge-)labelling?
or *label the edges, so that, "somehow", adjacent vertices get distinguished*

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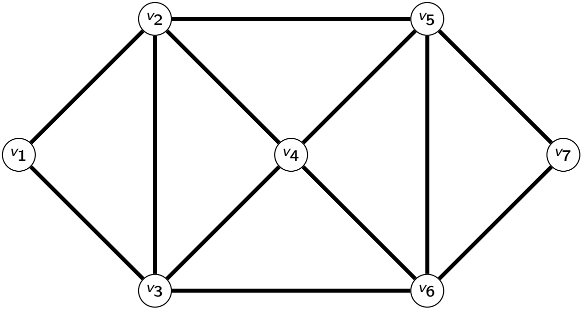

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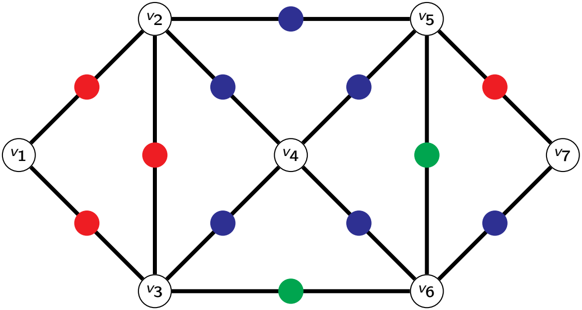

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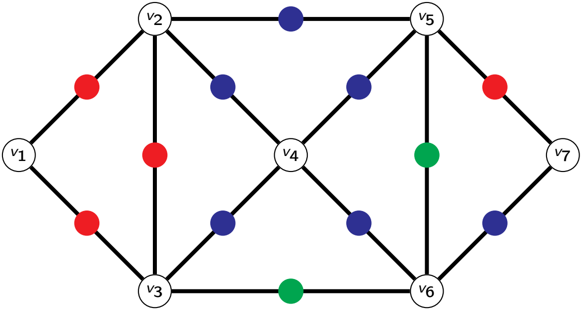

$\mathrm{Thing}(v_i)$:= Set of labels "incident" to v_i:

$$\mathrm{Thing}(v_1) = \{{\color{red}\bullet}\} \quad \mathrm{Thing}(v_2) = \{{\color{red}\bullet}, {\color{blue}\bullet}\} \quad \mathrm{Thing}(v_3) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$$

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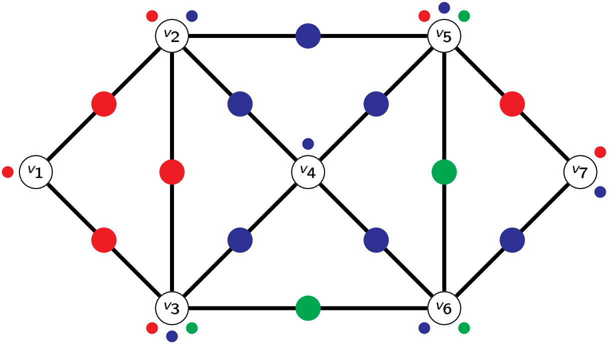

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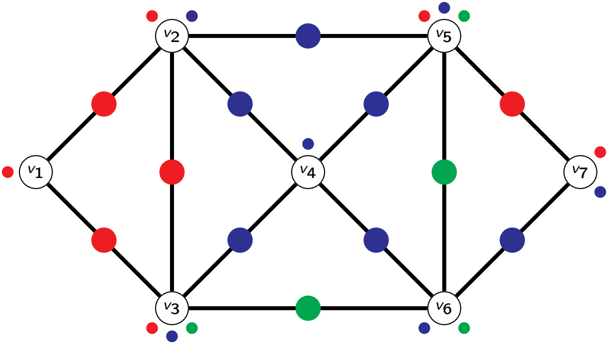

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Vertices are properly coloured ⇒ Neighbours can be distinguished!

From labels to colours

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- etc.

⇒ Dozens and dozens variants, with various applications/motivations...

A Dynamic Survey of Graph Labeling

Joseph A. Gallian
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University of Minnesota Duluth
Duluth, Minnesota 55812, U.S.A.
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Submitted: September 1, 1996; Accepted: November 14, 1997
Twentieth edition, December 22, 2017
Mathematics Subject Classifications: 05C78

Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 2500 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

The 1-2-3 Conjecture, in few words

Deals with **proper 3-labellings:**

- Labelled elements ⇒ Edges
- Label restrictions ⇒ Labels $1, 2, 3$, assigned improperly
- Distinguish what ⇒ Adjacent vertices
- Thing ⇒ Sum of incident labels

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"Given a graph, can we assign $1, 2, 3$ to its edges, so that no two adjacent vertices are incident to the same sum of labels?"

Edge weights and vertex colours

Michał Karoński and Tomasz Łuczak

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland
E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

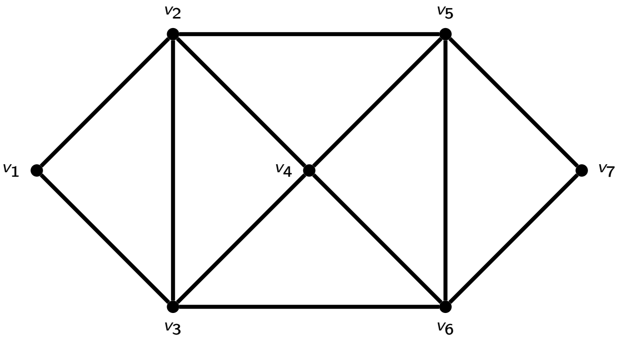
DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
E-mail: a.g.thomason@dpmms.cam.ac.uk

Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?

We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

Sample example



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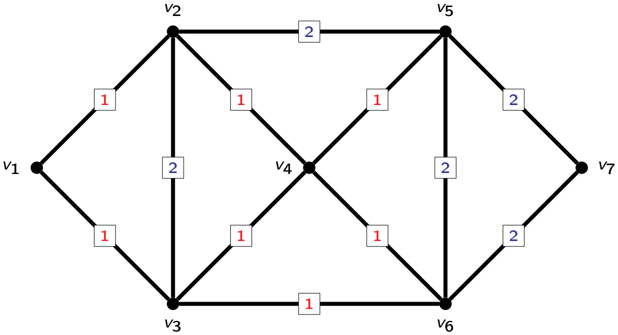

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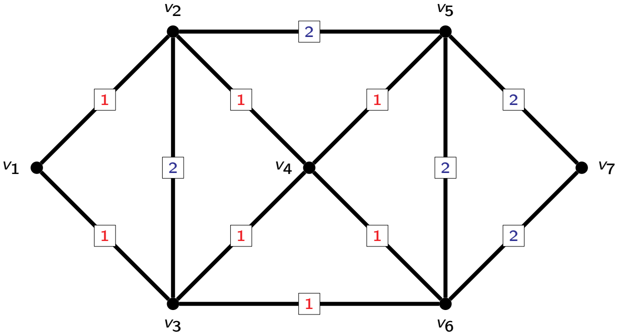

$\mathrm{Sum}(v_1) = 2$ $\mathrm{Sum}(v_2) = 6$ $\mathrm{Sum}(v_3) = 5$ $\mathrm{Sum}(v_4) = 4$

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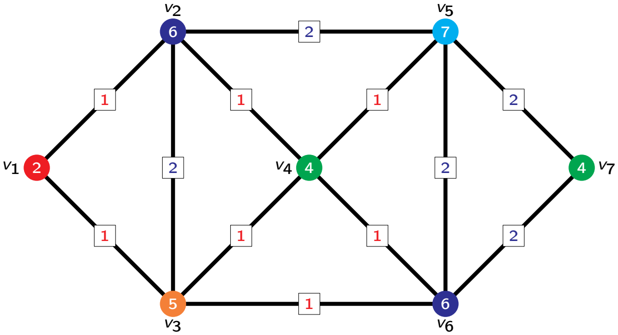

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More formal

G graph with k-labelling $\ell : E(G) \to \{1,\ldots,k\}$
$c_\ell : V(G) \to \mathbb{N}^*$ incident sums by ℓ
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1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

For every nice graph G, we have $\chi_\Sigma(G) \leq 3$.

Two interpretations/motivations...

... leading to different questions

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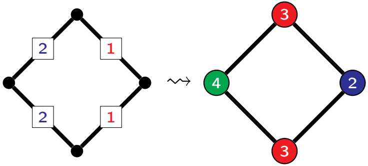

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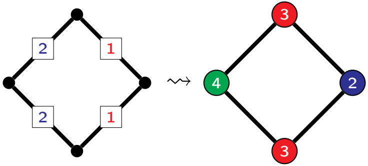

1-2-3 Conjecture

⇔

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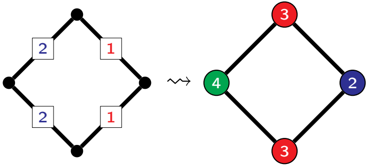

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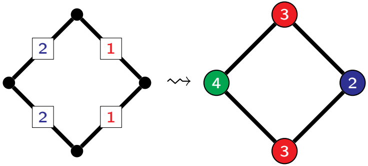

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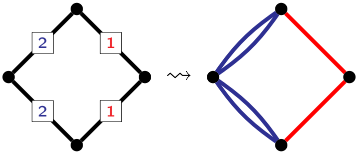

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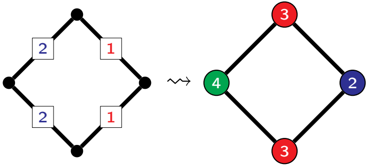

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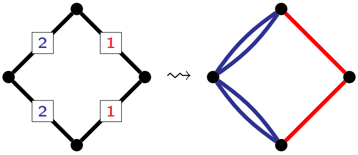

1-2-3 Conjecture

$\Leftrightarrow$

Make a graph locally irregular by multiplying every edge by at most 3

The 1-2-3 Conjecture's starter pack

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... + a few **parallel worlds**: product version, multiset version, etc.

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- **Results on main aspects:**
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– First focus –

Generalising the 1-2-3 Conjecture to digraphs

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Here, can we come up with an interesting digraph problem?

Want something "close" to the original 1-2-3 Conjecture:

- Somewhat natural — i.e., easy to define
- Similar behaviours — effects of labelling an edge, etc.
- A notion of exception — just as K_2
- Challenging — resilient to inductive arguments, etc.

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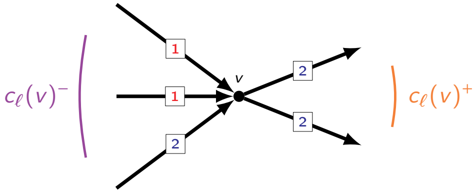

First thoughts

- Generalising graph problems to digraphs is natural...
- ... but the outcomes are uncertain!

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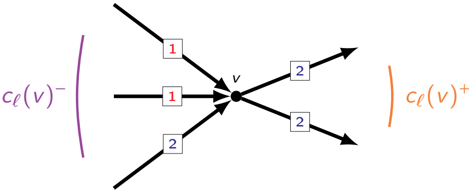

$\Rightarrow$ Many generalisation possibilities ☺ !

Previous works and general terminology

One earlier variant by Borowiecki *et al.*: distinguish via $|c_\ell^+ - c_\ell^-|$ (relative sum)

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- $-$ associated to c_ℓ^-, $+$ associated to c_ℓ^+
- for $\alpha, \beta \in \{-,+\}$, notions of (α,β)-proper labellings and (α,β)-nice digraphs
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$\Rightarrow$ Results in four "natural" variants

Results on the four "natural" directed variants

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moderate label influence

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Equivalent to $(+,+)$

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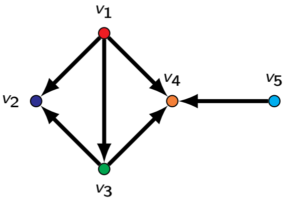
Labelling digraphs vs. Labelling bipartite graphs

Digraph $D \to$ Bipartite graph $B(D)$



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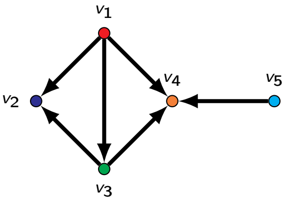

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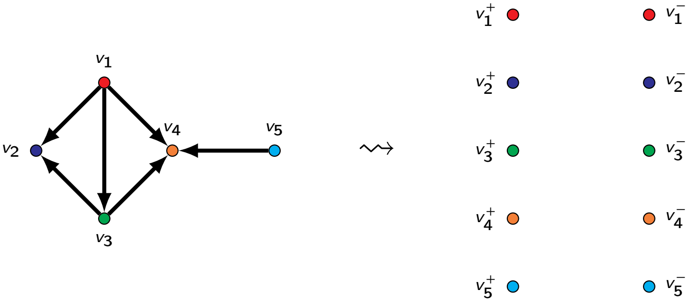

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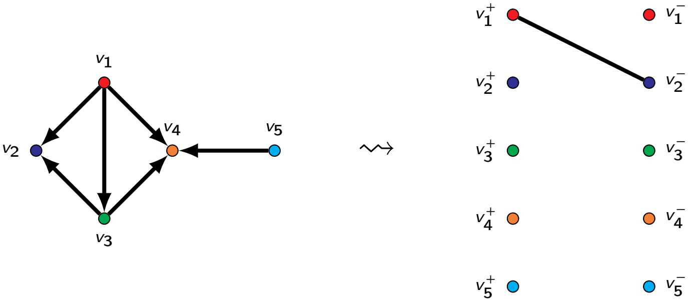

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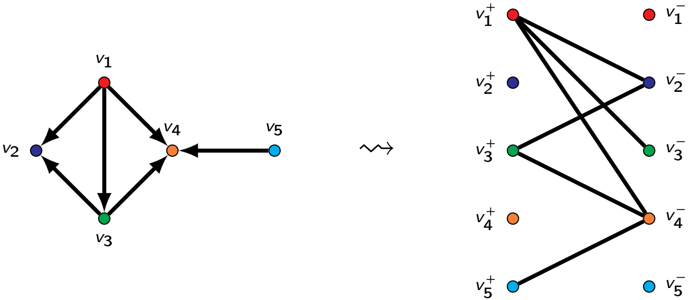

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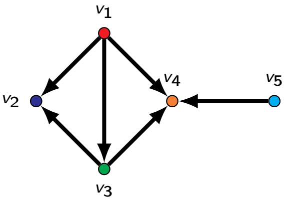

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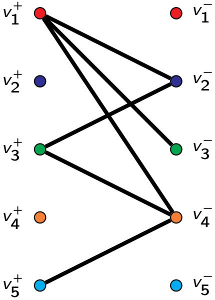

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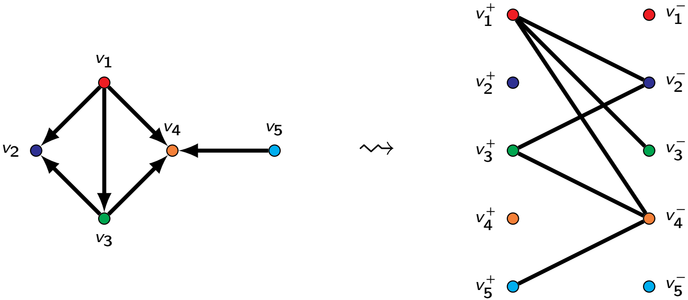

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- For a labelling ℓ of D and the derived one ℓ' of $B(D)$:

$$c_\ell^+(v) = c_{\ell'}(v^+) \text{ and } c_\ell^-(v) = c_{\ell'}(v^-)$$

  for every $v \in V(D)$

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Solving the $(-,+)$ variant

Theorem [B., Lyngsie, 2020]

Every nice connected bipartite graph G with bipartition $U \cup V$ has a proper 3-labelling ℓ where:

- for every $u \in U$, we have $c_\ell(u) \in \mathcal{U}$ and
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- $\mathcal{U} = \{0,3\} \cup \{3k+1 : k \geq 1\}$ and
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Proved through a bottom-to-top approach, considering a layer decomposition

Conclusion

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Is the "quest" over?

- What about playing with functions of c_ℓ^- and c_ℓ^+?
- Other parameters?

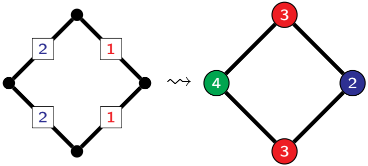
– Second focus –

The role of 3's for the 1-2-3 Conjecture

Two "natural" questions

Interpretation 1

Encode a proper vertex-colouring



1-2-3 Conjecture

⇔

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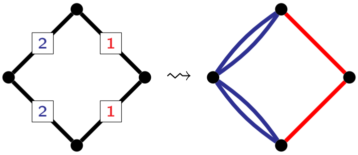
Make a graph locally irregular



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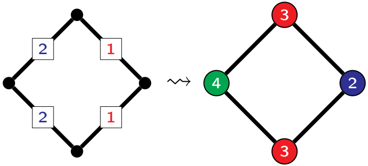

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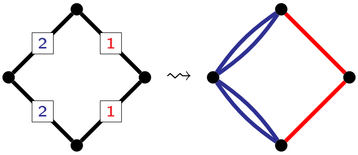

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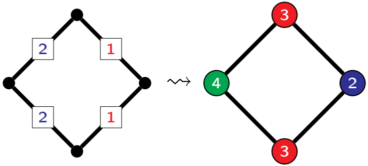

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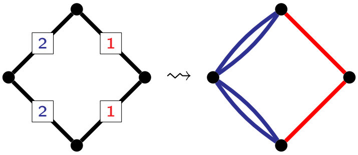

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How "small", i.e., in terms of number of edges, is the smallest locally irregular multigraph overlaid by G?

Note: We are not restricted to assigning labels $1, 2, 3$ only

Min. distinct colours mC

Min. sum of labels mL

Three optimisation problems over the 1-2-3 Conjecture

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$\boxed{\textbf{Min. maximum colour } \mathrm{mS}}$

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lower, upper bounds for classes
complexity results
etc.

"Presumption": $\mathrm{mS}(G) \leq 2\Delta(G)$

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"Presumption": $\mathrm{mL}(G) \leq 2|E(G)|$

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Note: We are not restricted to assigning labels 1, 2, 3 only

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Also observed in earlier works (1-2 Conjecture, Equitable 1-2-3 Conjecture, etc.)

A last problem

Terminology:

- $\mathrm{mT}(G)$: Minimum number of 3's in a proper 3-labelling of G
- $\mathscr{G}_p$: Graphs G with $\mathrm{mT}(G) = p$ $\qquad p \geq 0$
- $\mathscr{G}_{\leq p}$: $\mathscr{G}_0 \cup \cdots \cup \mathscr{G}_p$
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Questions:

- Given $\mathscr{F}$, is there $p \geq 0$ such that $\mathscr{F} \subset \mathscr{G}_{\leq p}$?
- If not, how can we bound $\rho(\mathscr{F})$ above?

Obtained results

- **Relevance of the problem:**
  - Every $\mathscr{G}_p$ is well populated... $\forall H, \exists G : H \leq_i G$ and $G \in \mathscr{G}_p$
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Two basic ideas:

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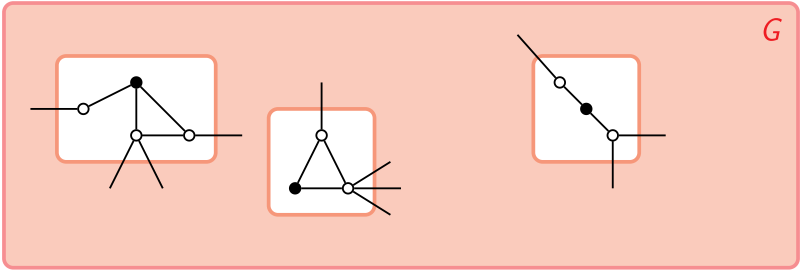

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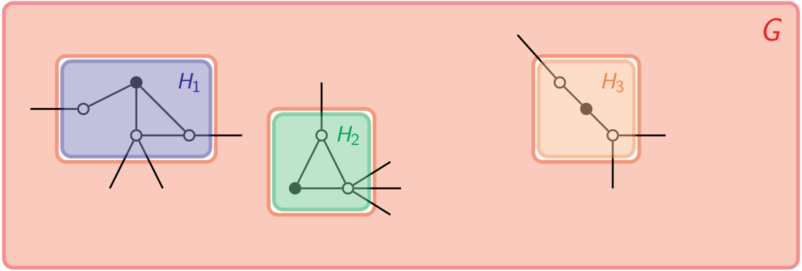

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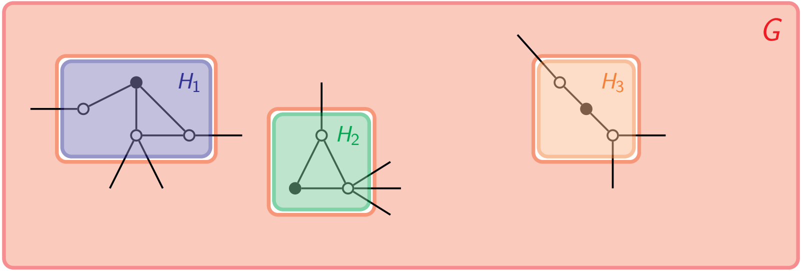

Want: $\mathrm{mT}(G) \geq \mathrm{mT}(H1) + \mathrm{mT}(H_2) + \mathrm{mT}(H3)$

Weakly induced subgraphs

Observation [B., Fioravantes, Mc Inerney, 2020+]

Let H be a graph, and G be a supergraph of H where $d_H(v) = d_G(v)$ for every $v \in V(H)$ with $d_H(v) > 1$. Then, $\mathrm{mT}(G) \geq \mathrm{mT}(H)$.

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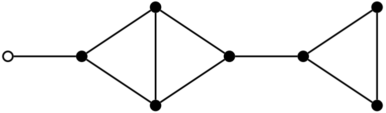
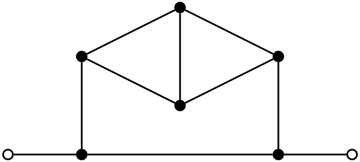
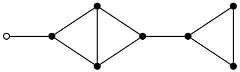
Such graphs H exist, with low ratio ρ(H) (1/10)!





Example application



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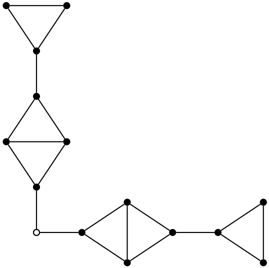

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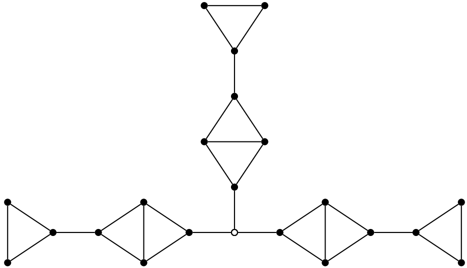

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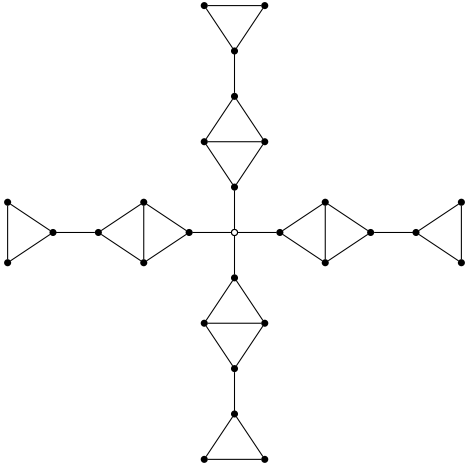

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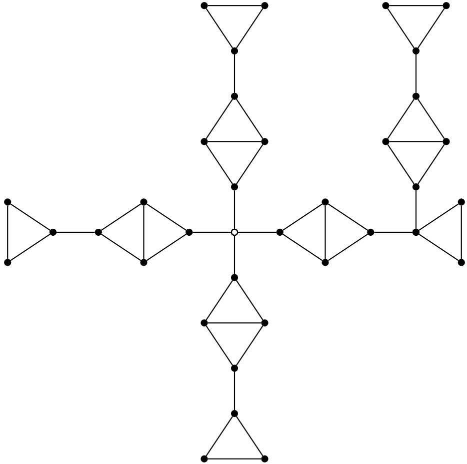

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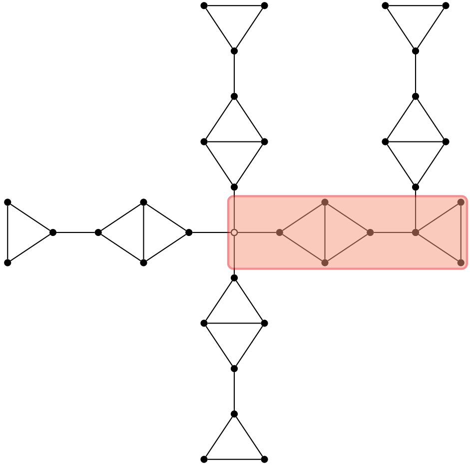

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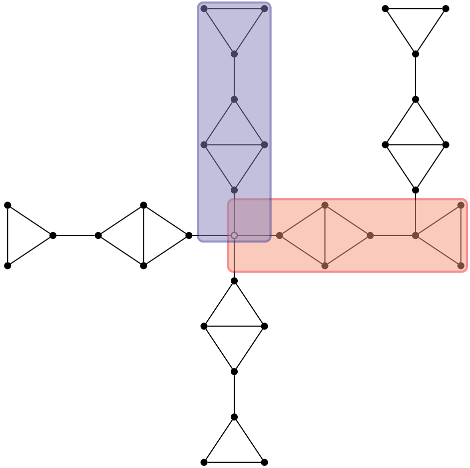

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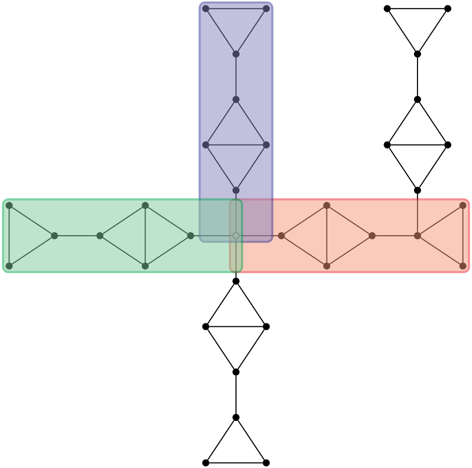

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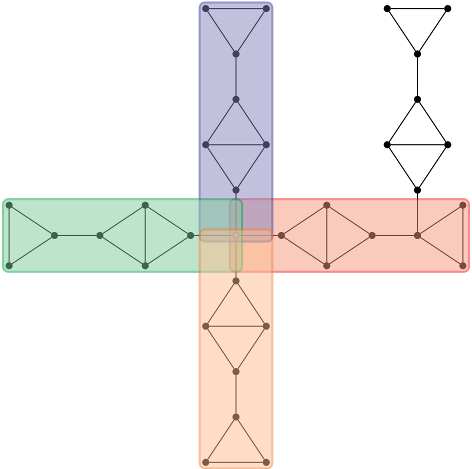

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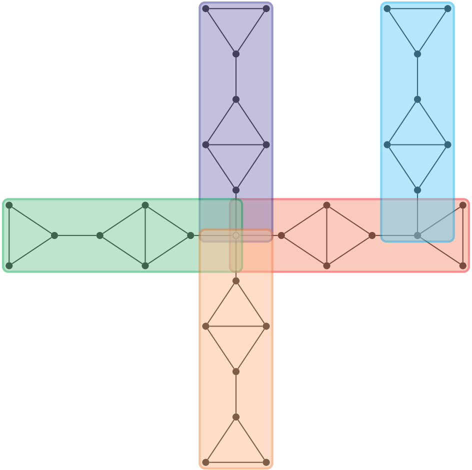

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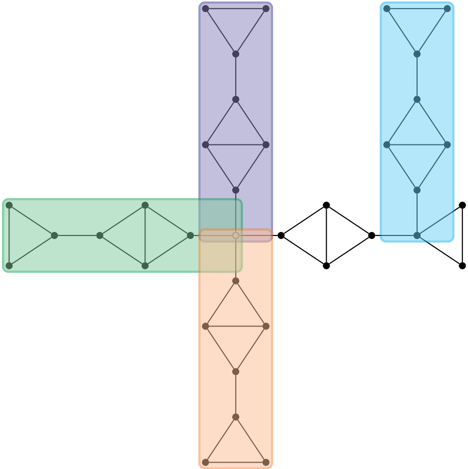

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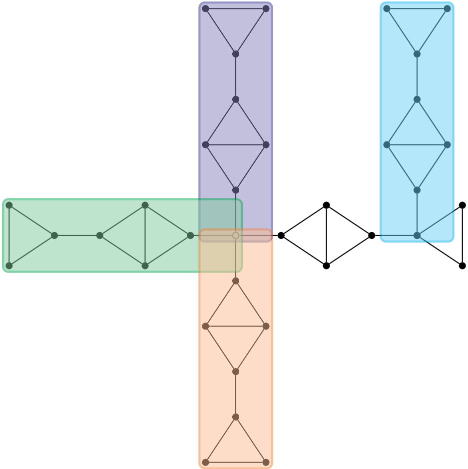

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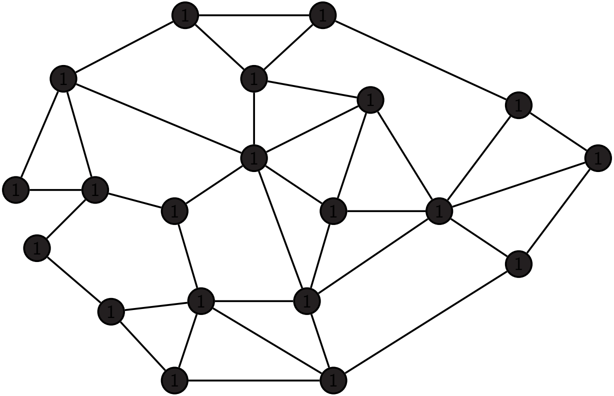
Use "nice" pieces, combined cleverly, to get more general properties

Establishing upper bounds for 3-chromatic graphs

Standard approach: Proper 3-labellings with distinct colours modulo 3



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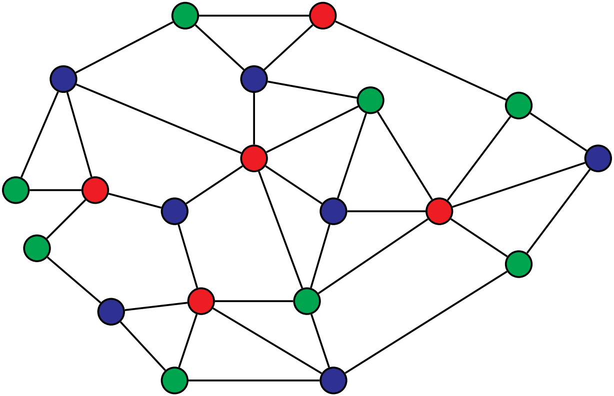

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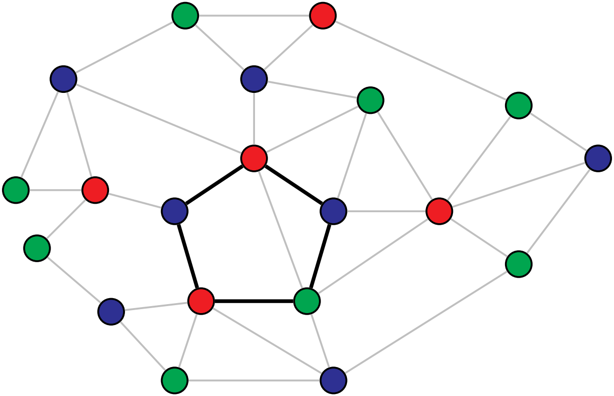

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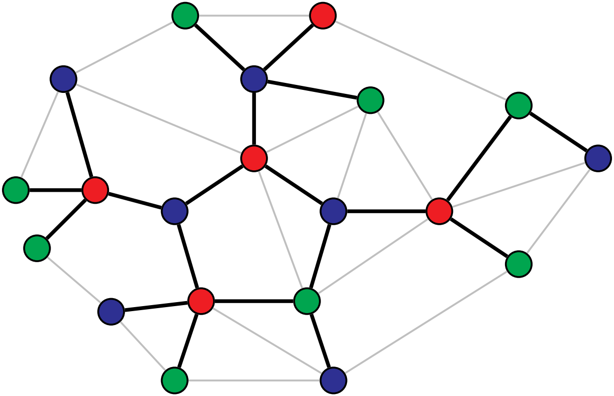

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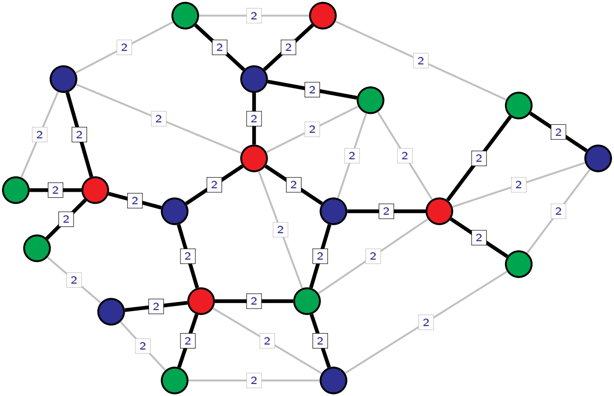

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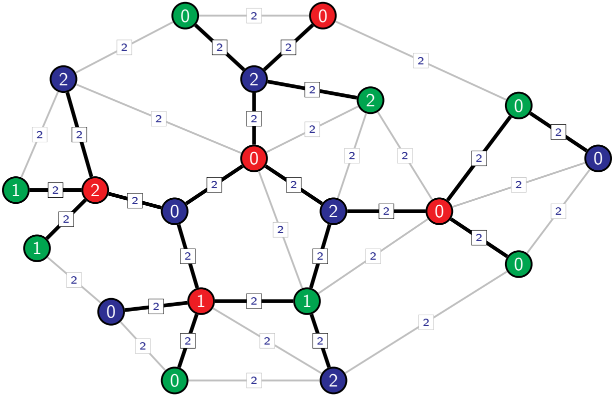

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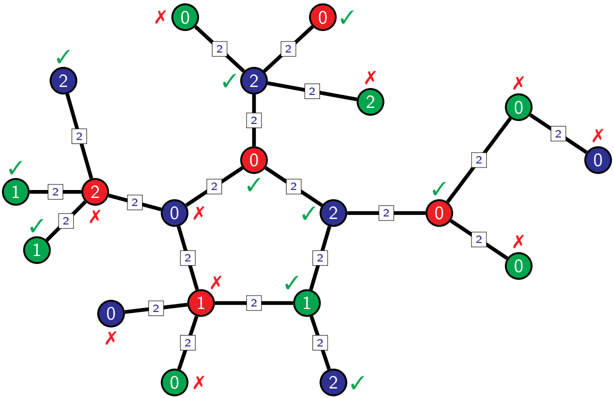

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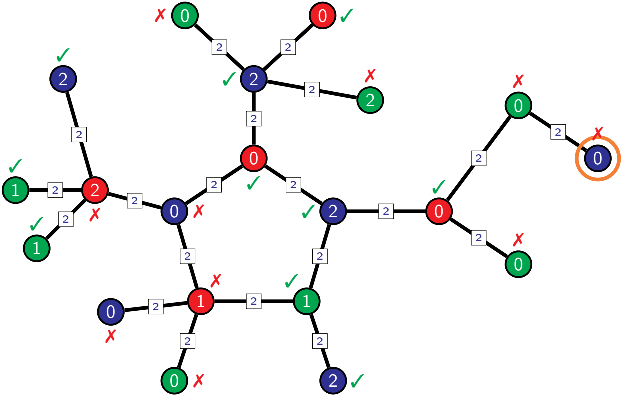

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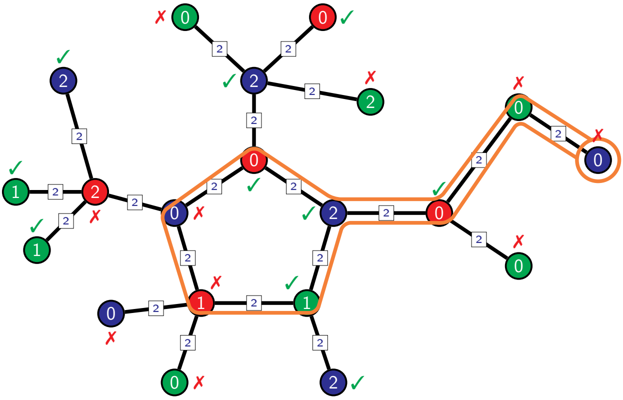

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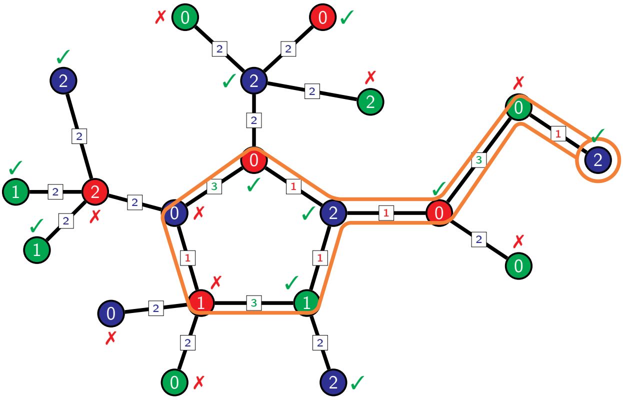

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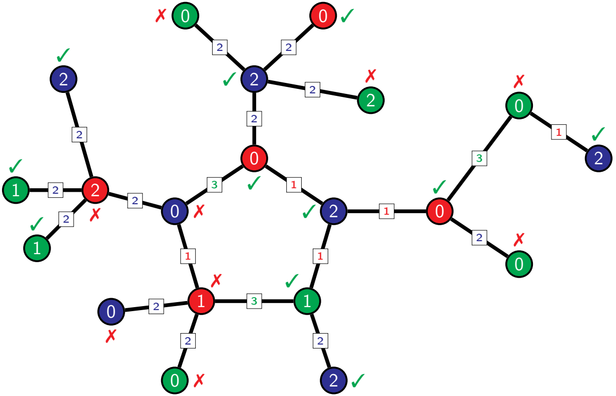

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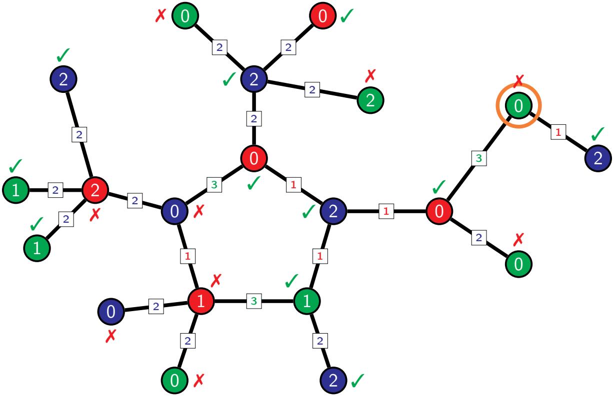

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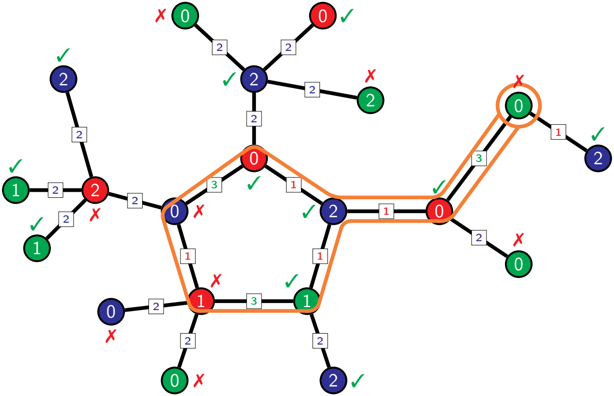

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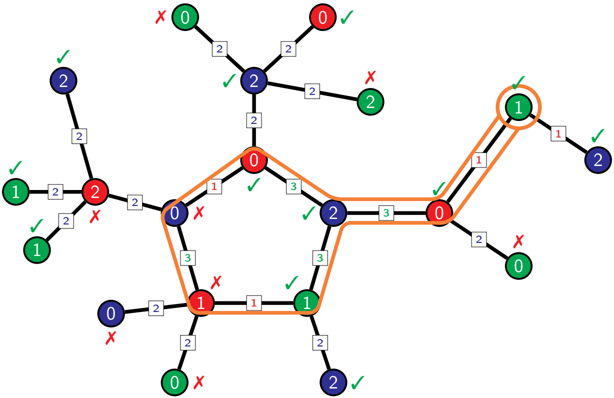

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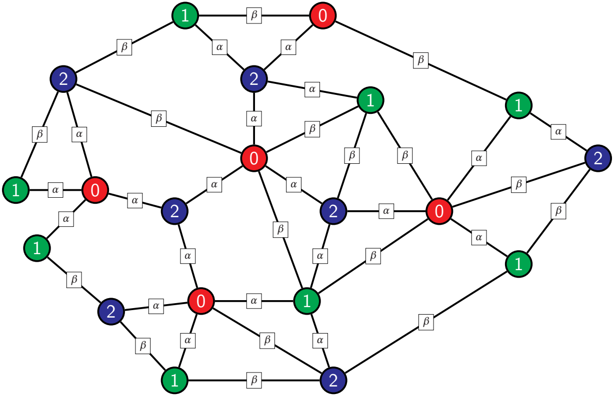

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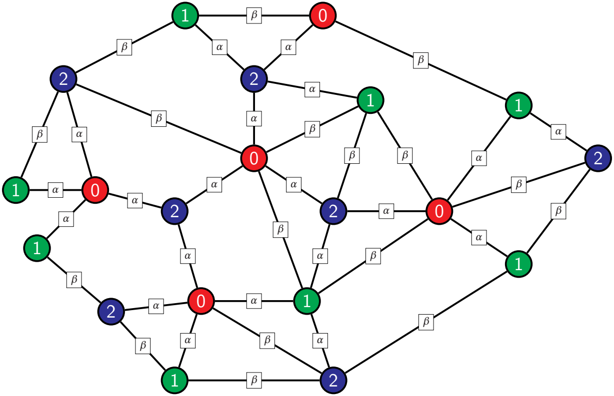

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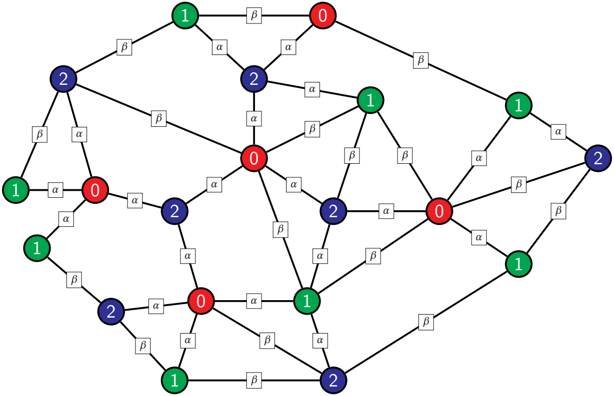

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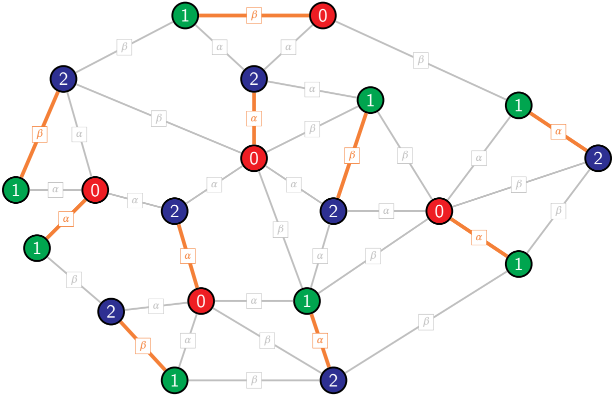
Going farther



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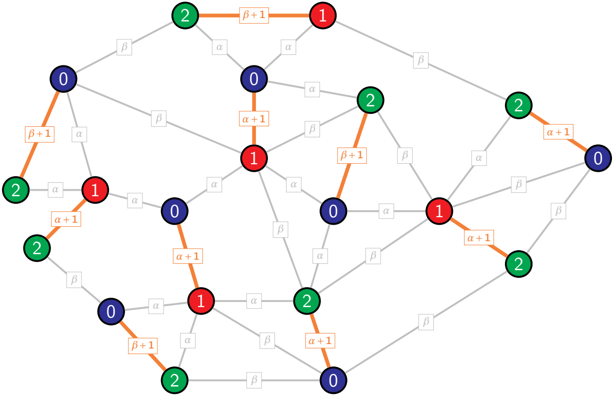

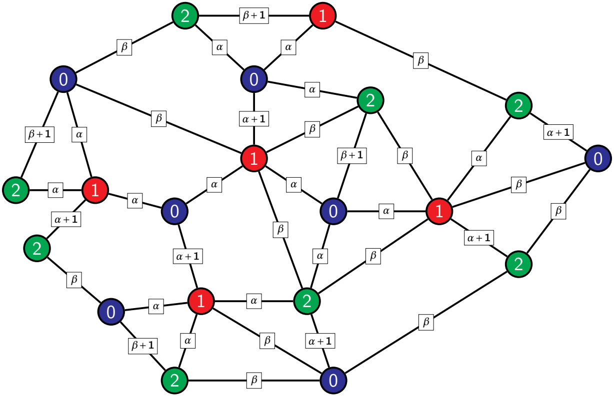

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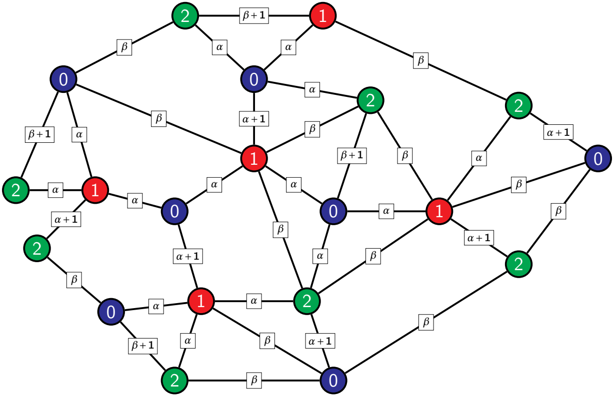
Going farther



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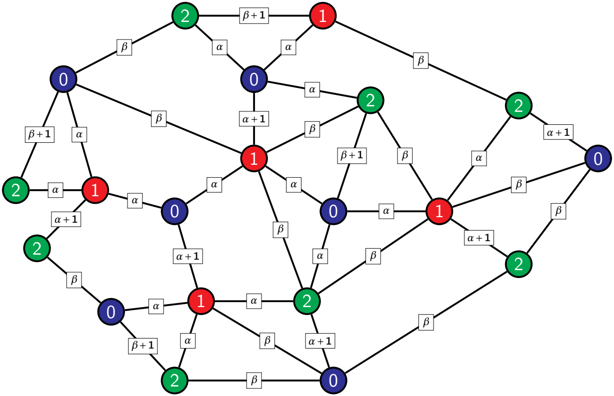

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- Can be repeated on several spanning subgraphs!
- $\Rightarrow$ perfect matchings, cycle covers (Hamiltonian cycles), etc.

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- ... but rather hard to study in general ☹
- Limits due to restricted knowledge of the 1-2-3 Conjecture

Some conclusions and perspectives

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  - **Verify the 1-2-3 Conjecture for more graphs**
    - Planar graphs, 4-chromatic graphs, regular graphs
    - Etc.

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    - Etc.
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Thank you for your attention!

Deliberations in progress...

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