On partitioning graphs into connected subgraphs

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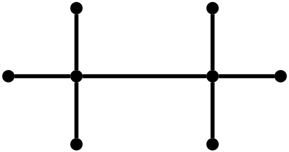

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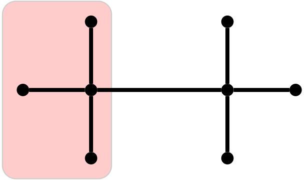

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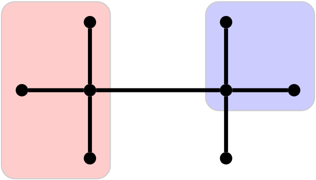

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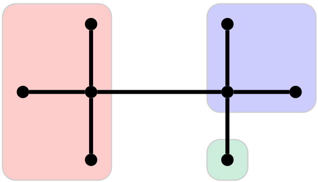

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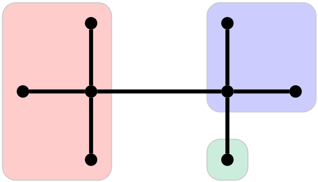

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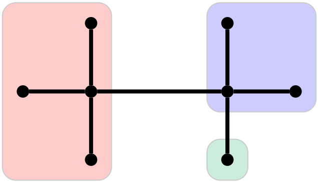

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$(V_1, \ldots, V_p)$ is a **realization** of $(n_1, \ldots, n_p)$ in G.

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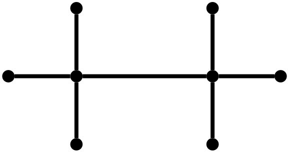

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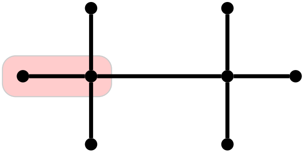

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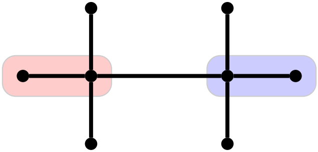

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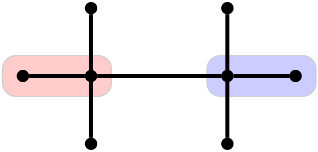

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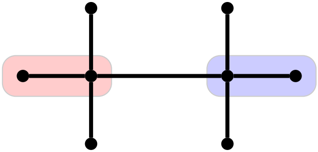

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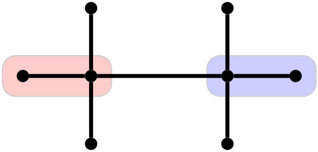

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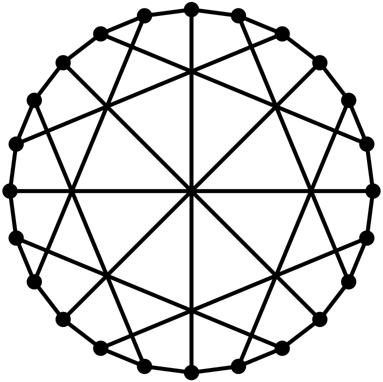

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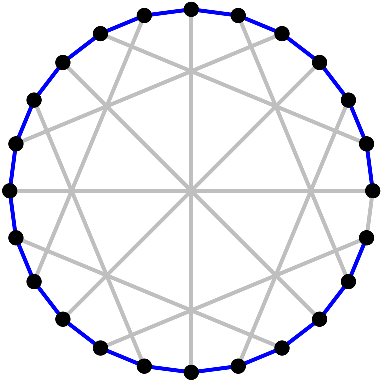

Considered aspects

So far, considered aspects of AP graphs include:

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Many open questions...

Algorithmic aspects

Complexity of partitioning a graph

"Atomic" decision problem:

REALIZATION
Input: A graph G, and a partition $\pi := (n_1, ..., n_p)$ of $|V(G)|$.
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So, what about the problem $\mathrm{AP} = \{$Graph G: is G AP?$\}$?

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$\Rightarrow$ Generally yield checking algorithms.

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What for other classes of graphs?

(e.g. general trees, 3-connected near-triangulations, etc.)

Structural aspects

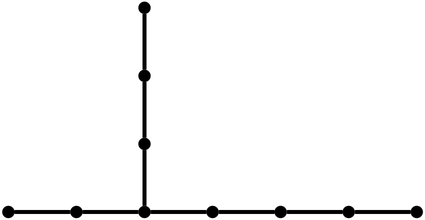
On AP trees

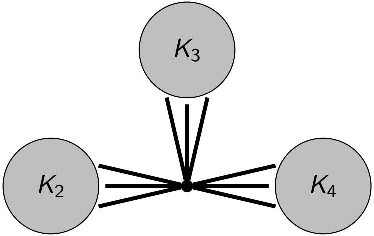
AP trees are rather understood:

Theorem [Barth, Fournier, Ravaux, 2009]

- AP trees have $\Delta \leq 4$;
- degrees at least 3 are located on a same path;
- degree-4 vertices are adjacent to a leaf.

Obtained by considering surgraphs that are "easier" w.r.t. the AP property:





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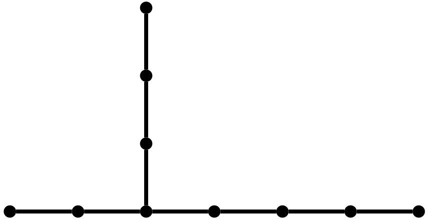

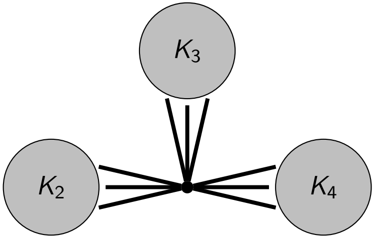

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So, actually:

Corollary [Barth, Fournier, Ravaux, 2009]

Removing a cut-vertex from an AP graph results in at most 4 components.

Generalization

Via the same technique:

Theorem [Baudon, Foucaud, Przybyło, Woźniak, 2014]

For any $k \geq 2$, removing a k-cutset from an AP graph:

- may result in arbitrarily many components,
- whose orders grow exponentially.

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Theorem [Ore, 1960]

Let G be a graph with order n. If for every two non-adjacent vertices u and v of G we have $d(u) + d(v) \geq n - 1$, then G is traceable.

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Let G be a graph with order $n \geq 8$. If $\alpha(G) \leq \lceil n/2 \rceil$ and for every two non-adjacent vertices u and v of G we have $d(u) + d(v) \geq n - 3$, then G is AP.

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Brandt claimed a generalization to triples of independent vertices.

Hamiltonicity via edge density

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Theorem [Folklore?]

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Theorem [Kalinowski, Pilśniak, Schiermeyer, Woźniak, 2016]

Let G be a connected graph with order $n \geq 22$. If $|E(G)| > \binom{n-4}{2} + 12$, then G is AP.

Squares of graphs

Well-known result:

Theorem [Fleischner, 1976]

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... also wrong for AP graphs:

Theorem [B., Li, 2018+]

Realization is NP-complete, even when restricted to squares of bipartite graphs.

Forbidding pairs of patterns

Another well-known result:

Theorem [Duffus, Gould, Jacobson, 1982]

Every 2-connected (resp. connected) $\{K_{1,3}, Z\}$-free graph is Hamiltonian (resp. traceable).

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For APness, none of the two patterns can be dropped from the equation:

Theorem [B., Li, 2018+]

REALIZATION is NP-complete, even when restricted to claw-free graphs, or to net-free graphs.

Further directions

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- Does hypotraceability imply APness?
- For $k \geq 3$, k-connected k-regular are not all traceable...
- ... what for AP graphs? [Diwan, 2003]
- Pick your favourite result on traceability. Does it weaken to AP graphs?

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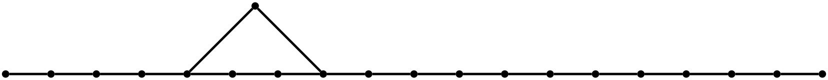

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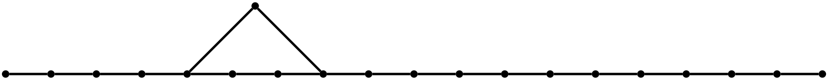

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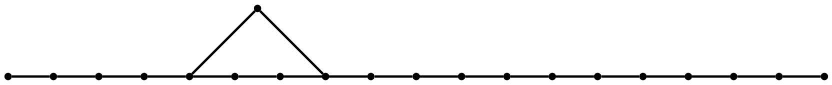

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Properties of minimal AP graphs?

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Known stuff:

- Largest known families: $m = \frac{31n}{30}$ [Baudon, Przybyło, Woźniak, 2012].

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- Largest known families: $m = \frac{31n}{30}$ [Baudon, Przybyło, Woźniak, 2012].
- If G minimal AP with $n \geq 6$, then $\Delta(G) \leq n - 3$ [B., 2014].

Progress so far

Not much known. Main conjecture:

Conjecture [Ravaux, 2009]

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Questions:

- Denser families?
- Generalization of the Δ property.
- Clique number?
- Families with connectivity $k \geq 2$?
- etc.

Perspectives, problems, etc.

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Thanks for your attention.