Sequential Metric Dimension (in trees)

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Introduction to the problem

Rules:

- Graph $G = (V, E)$;
- "Secret" vertex $t \in V$;
- Probing a vertex $v \Rightarrow \mathrm{dist}_G(v, t)$.

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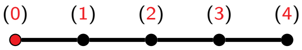

$\mathrm{MD}(P_n) = 1$

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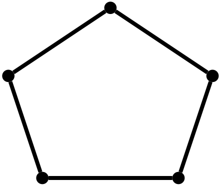

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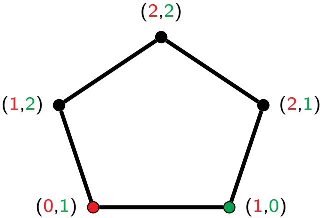

$\mathrm{MD}(C_n) = 2$

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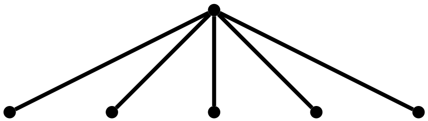

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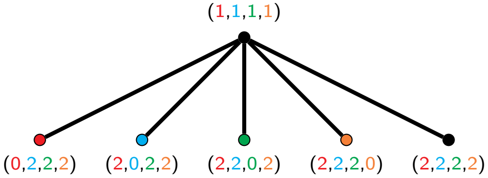

$$\mathrm{MD}(S_n) = n-1$$

How faster can we locate the target through *multiple* probing steps of *one* vertex?

Min. # of steps = Sequential Location Number $\mathrm{SL}(G)$ of G.

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$\sim$ related to a well-known game ☺ ...

Sequential Locating Game (Seager, 2013)

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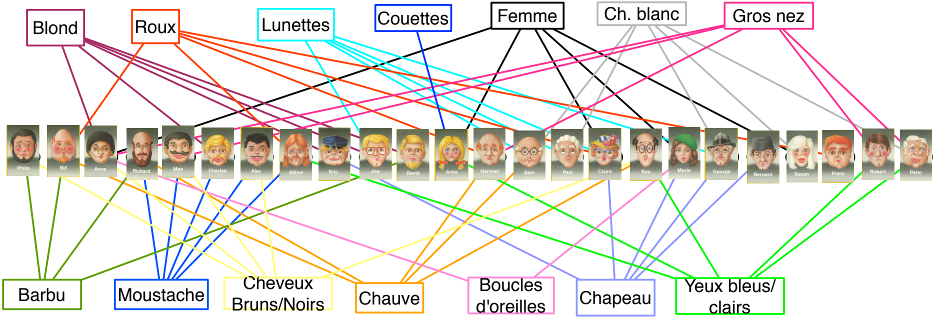
Sequential Locating Game and Guess Who?



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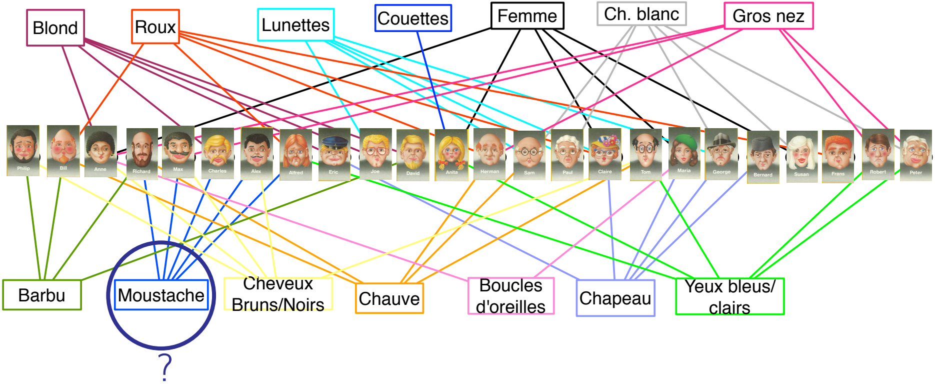

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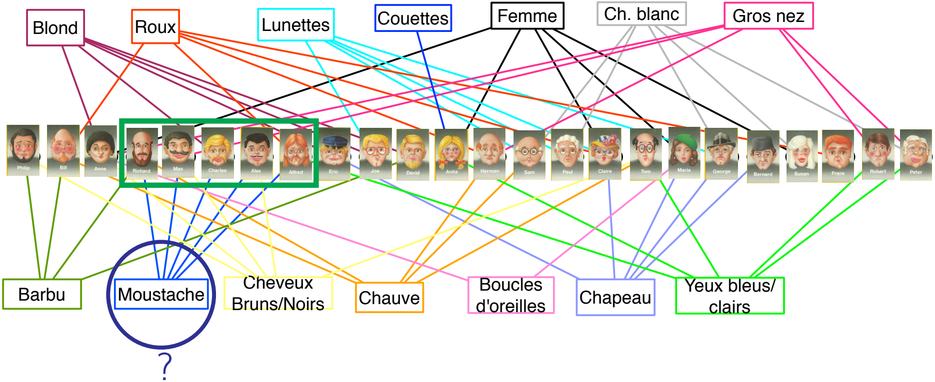

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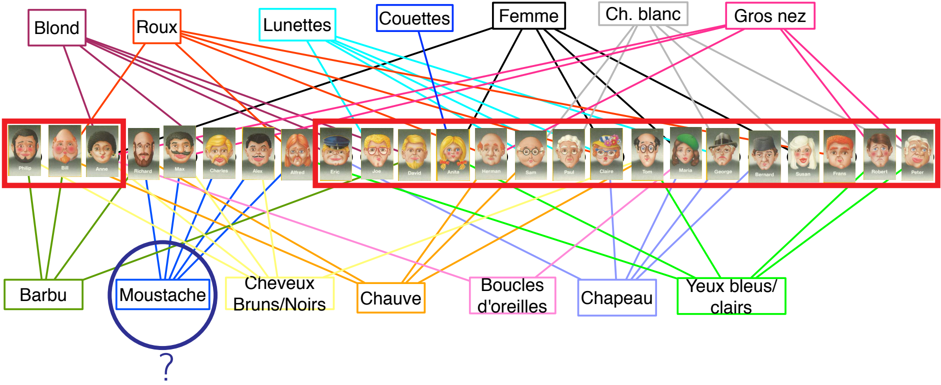

Question

Given G, k, ℓ, is it possible to locate an immobile invisible target in G in at most ℓ steps, by probing at most k vertices each step?

Related:

- $\ell = 1$ (Metric Dimension);
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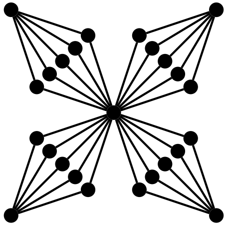

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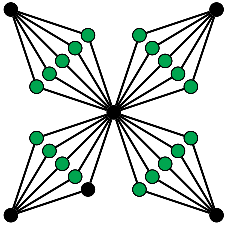

$\mathrm{MD}(G) = 19$

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Now sequential probing ($k = 4$)

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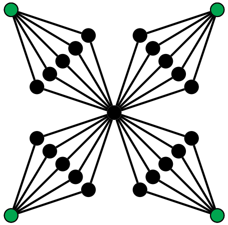
Location in 2 steps; FIRST step:



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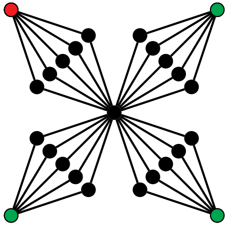

Answer: (0,4,4,4)

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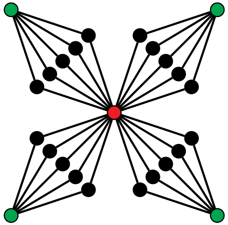

Answer: (2,2,2,2)

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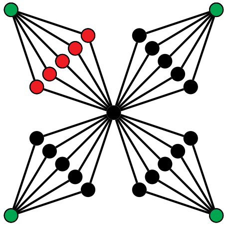

Answer: (1,3,3,3)

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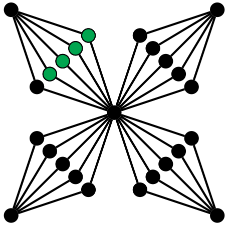

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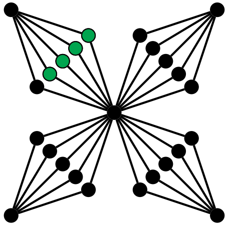

$\Rightarrow \mathrm{MD}(G) = 19$. And $\lceil \mathrm{MD}(G)/4 \rceil = 5$, while 2 steps suffice.

Sequential Metric Dimension in trees

$\lambda_k(T)$: min. # of steps to locate t in T (probing at most k vertices each step).

Localisation Problem

Given a tree T, can we locate an immobile invisible target in at most ℓ steps, provided we can probe at most k vertices each step?

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Localisation Problem

Given a tree T, can we locate an immobile invisible target in at most ℓ steps, provided we can probe at most k vertices each step?

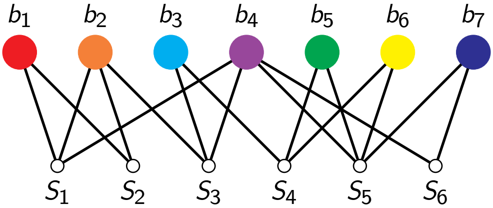
Tree case (k fixed, minimize ℓ):

- Making the appropriate first probing step is NP-complete ☹ ...
- ... but deciding how to probe optimally afterwards is polytime doable ☺ .
- ⇒ Polytime (+1)-approximation algorithm, yielding $\lambda_k(T)$ or $\lambda_k(T)+1$.

Theorem

Determining $\lambda_k(T)$ is NP-hard in trees T.

Proof (sketch). Reduction from Hitting Set (given a set $B := \{b_1, \ldots, b_n\}$ and a set $\mathscr{S} := \{S_1, \ldots, S_m\}$ of subsets of B, find a smallest subset of B hitting all S_i's).



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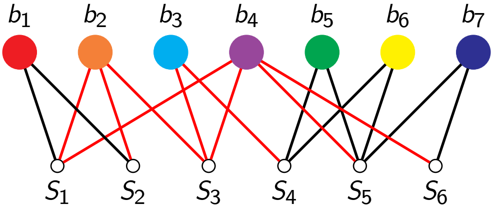

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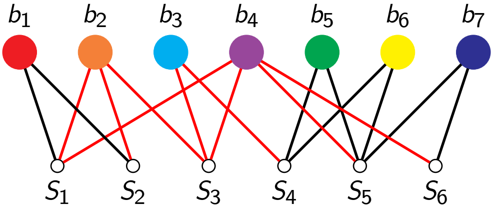

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- Have many big stars in the tree, so that the target has to hide in one such.
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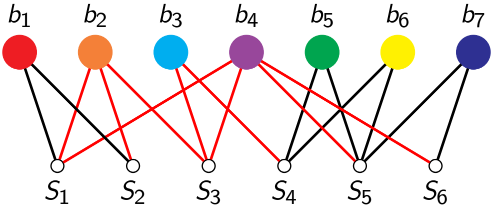

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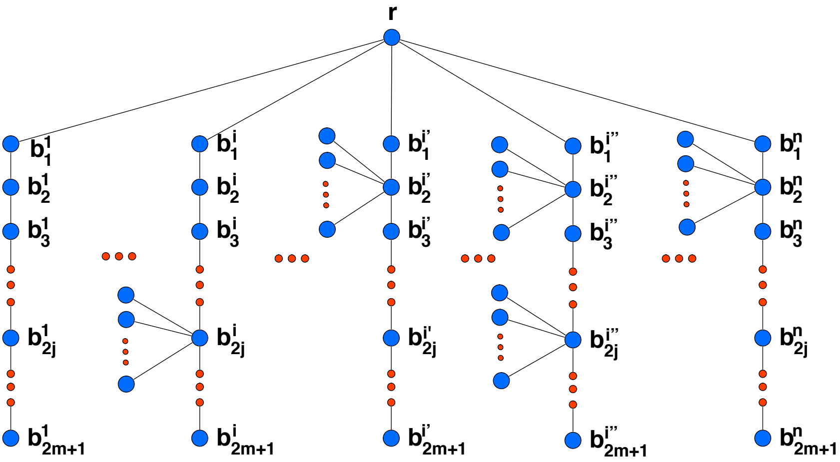
$\Rightarrow$ Identifying the big star early $\Leftrightarrow$ Hitting set. ■

Reduction, illustrated



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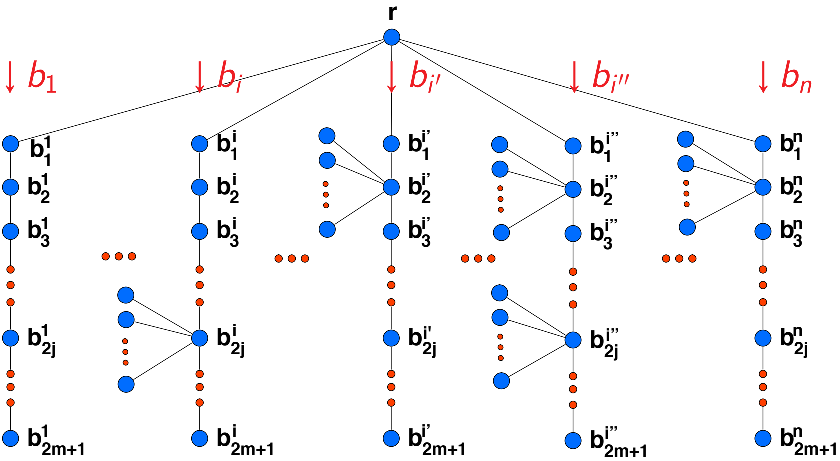

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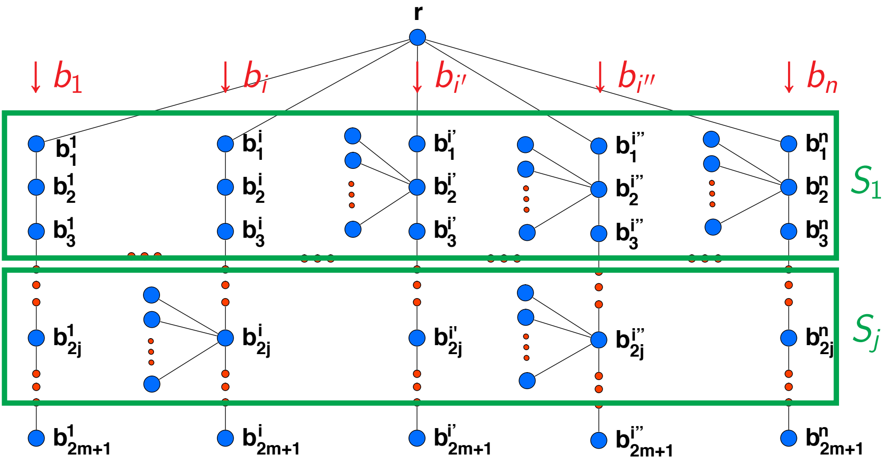

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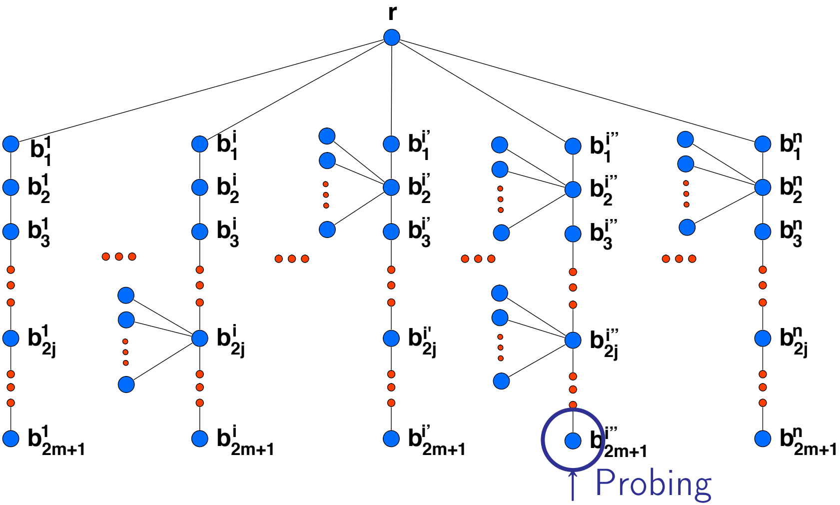

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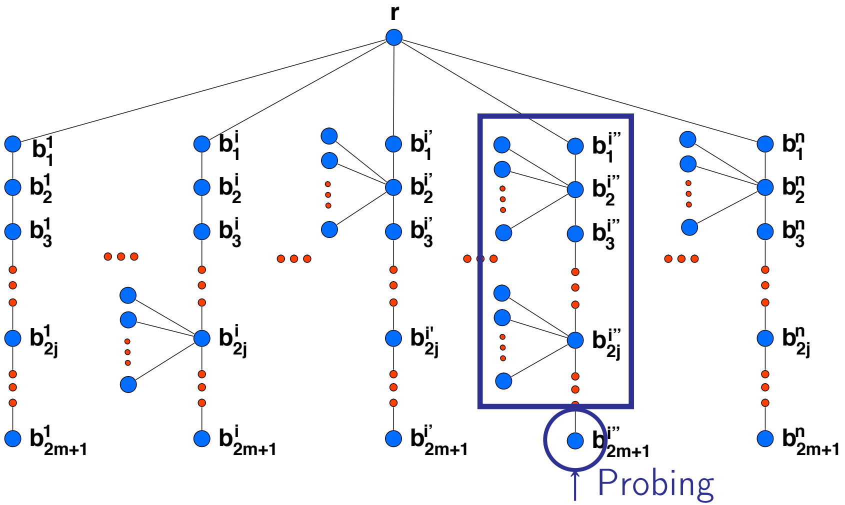

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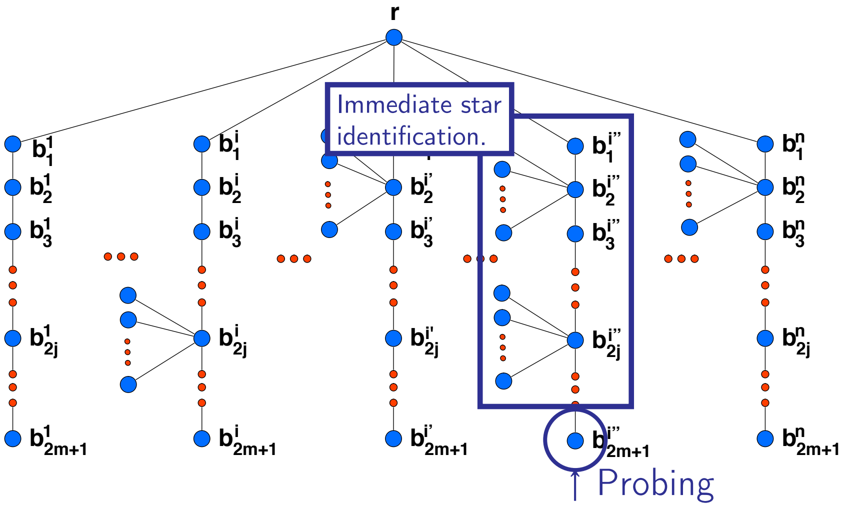

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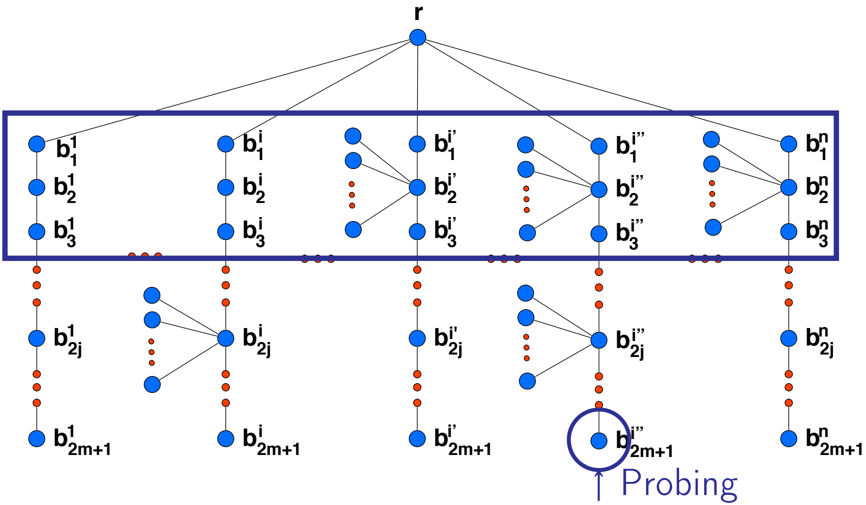

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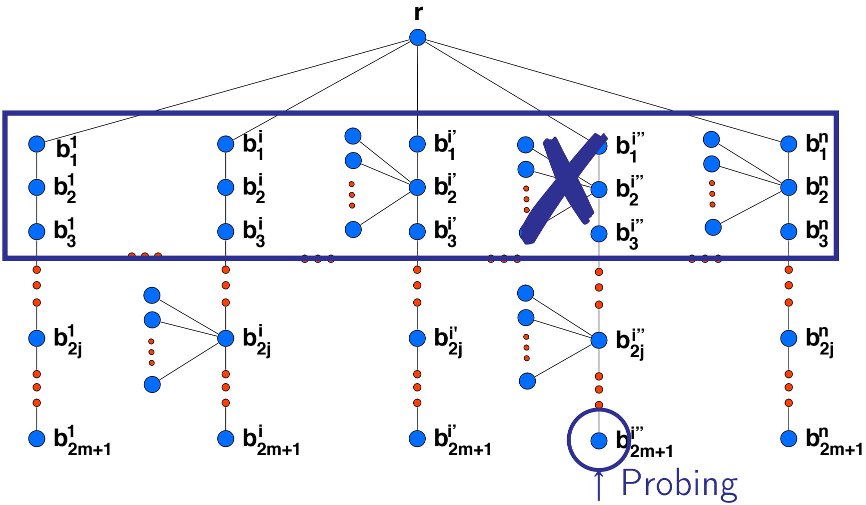

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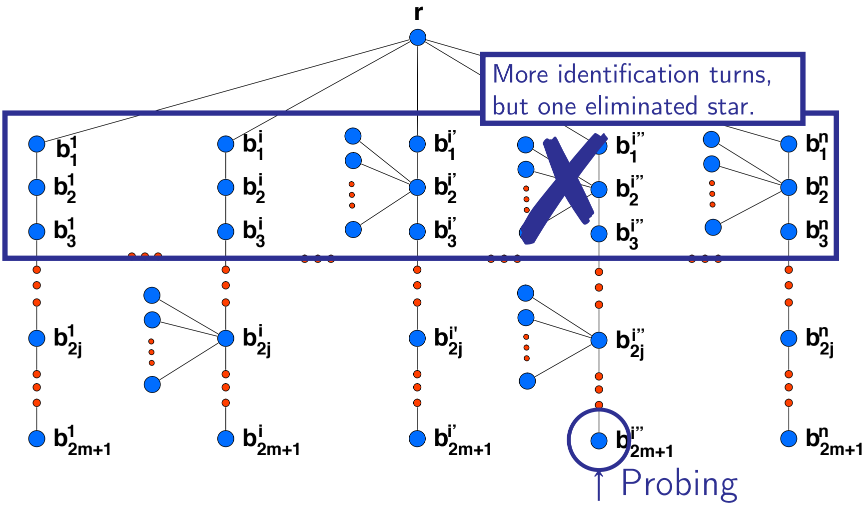

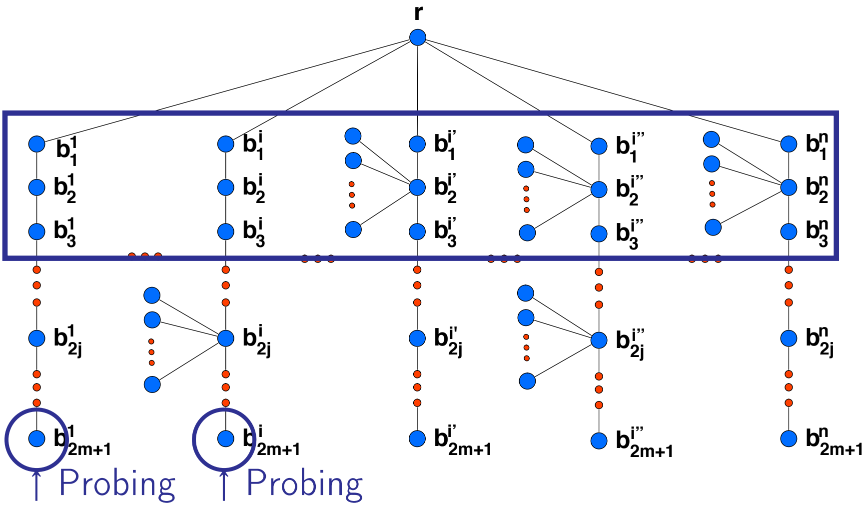
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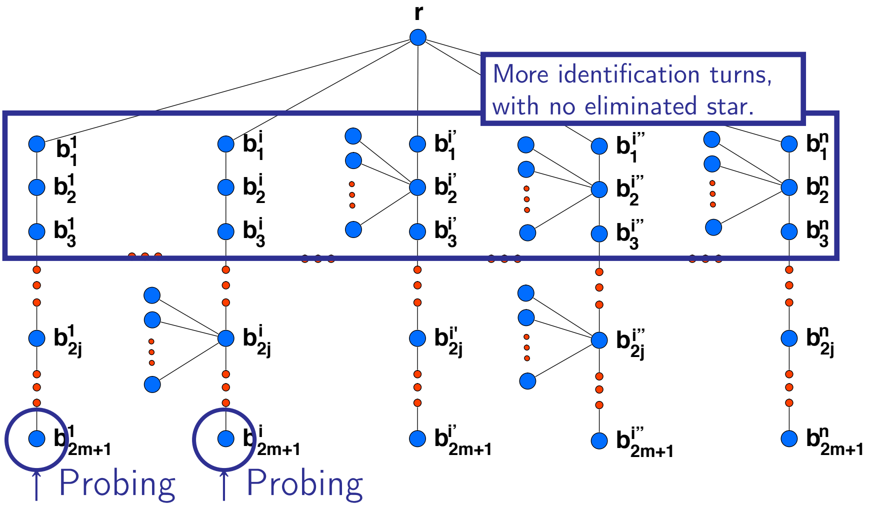
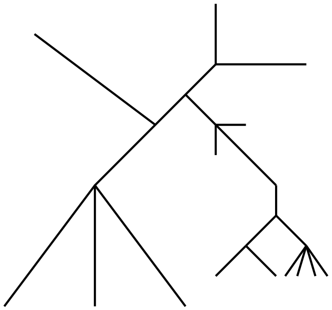


Want: First turn such that all S_i's are hit... ⇒ Hitting set.

First step: Probe any one vertex r...



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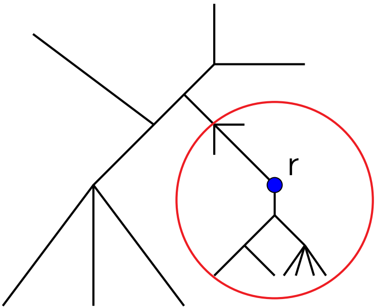

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- a tree T' rooted in r;
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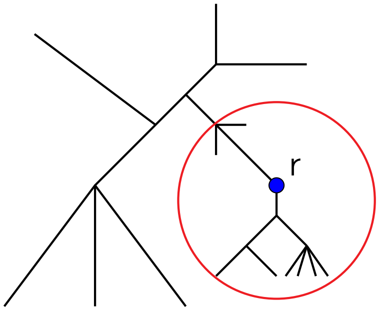

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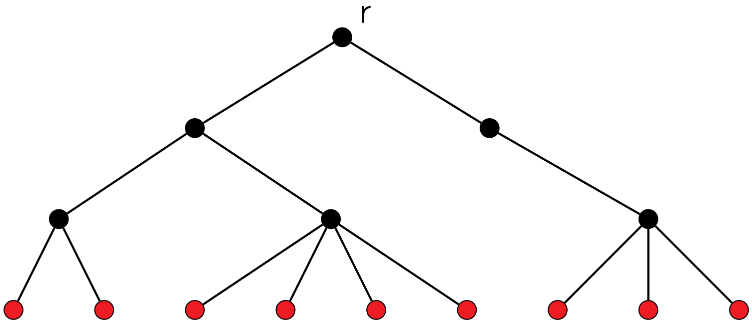

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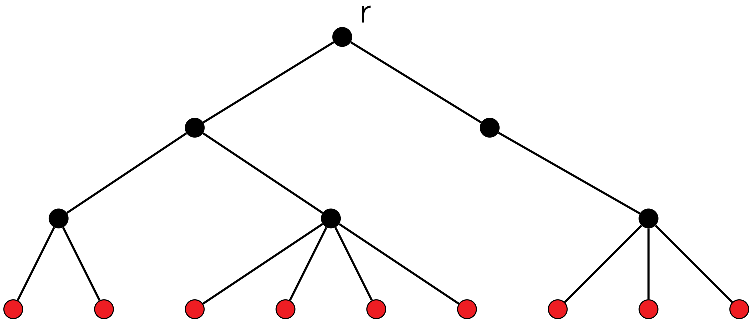
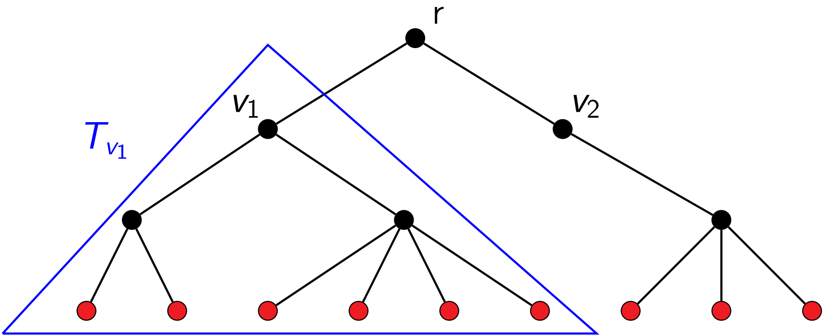


(because playing outside T' is pointless)

T_v: subtree rooted in v of T' rooted in r (v is a child of r).



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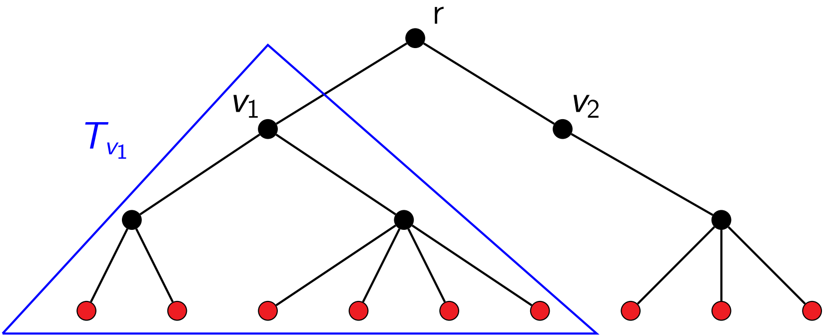

Key fact: Probing any vertex of T_v $\Rightarrow$ Know whether T_v hosts the target!

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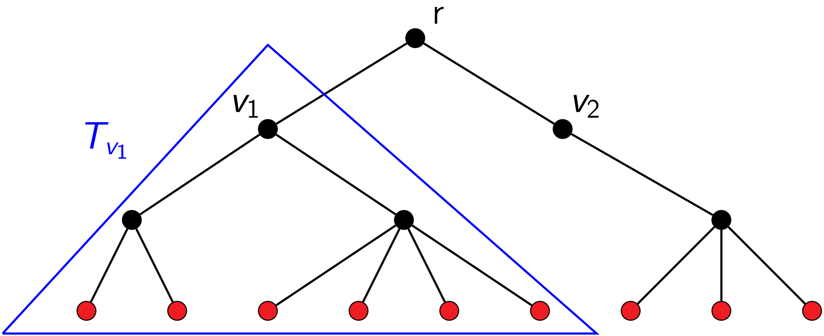

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Crucial question

When playing in T_v for the first time, how many vertices should be probed?

Example: What if T_{v_1} is a big star, while $T_{v_2}, ..., T_{v_{1000}}$ are much smaller stars?

$\Rightarrow$ Trade-off between inspecting **many** of the T_v's, and inspecting **efficiently**.

⇒ Trade-off between inspecting **many** of the T_v's, and inspecting **efficiently**.

Two main parameters for each T_v (assuming target on a leaf):

1. $\lambda_k(T_i)$: min. # of steps needed, probing at most k vertex each step;
2. $\pi_k(T_i)$: min. # of vertices that can be probed **during the first step** in a strategy taking $\lambda_k(T_i)$ steps.

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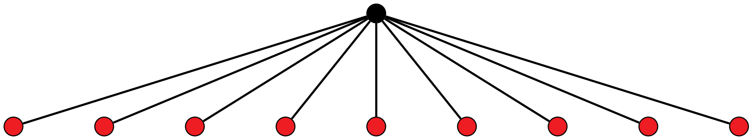

⇒ Trade-off between inspecting **many** of the T_v's, and inspecting **efficiently**.

Two main parameters for each T_v (assuming target on a leaf):

1. $\lambda_k(T_i)$: min. # of steps needed, probing at most k vertex each step;
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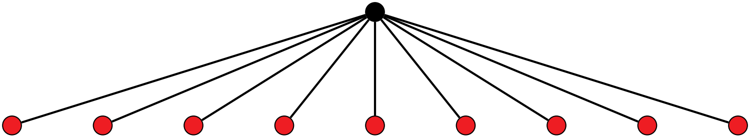

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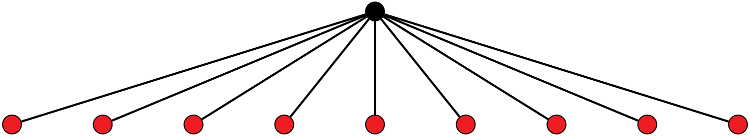
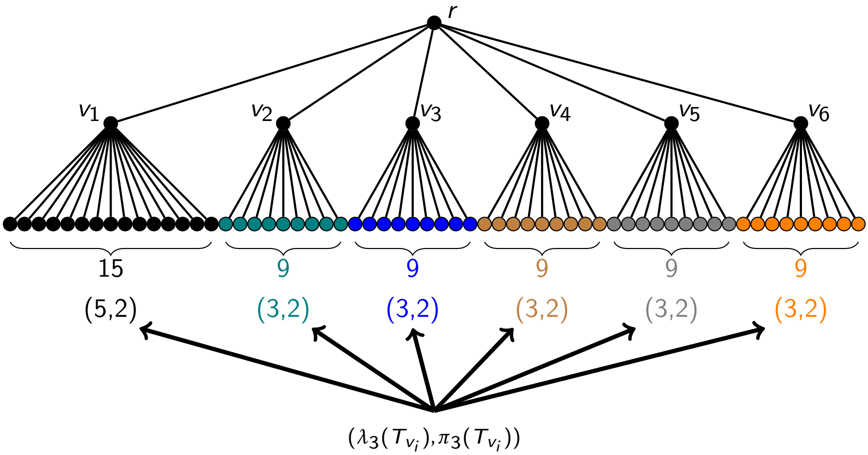


$\lambda_3(T) = 3$ $\pi_3(T) = 2$

In next example, $\lambda_3(T') = 5$.



Full example

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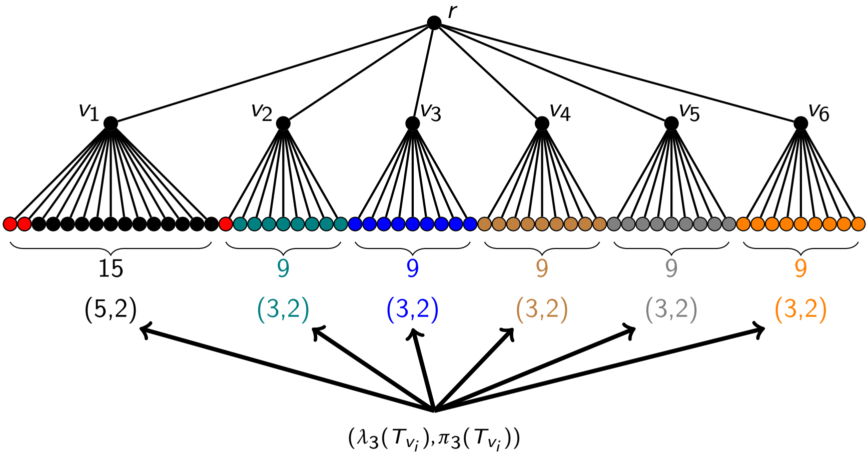

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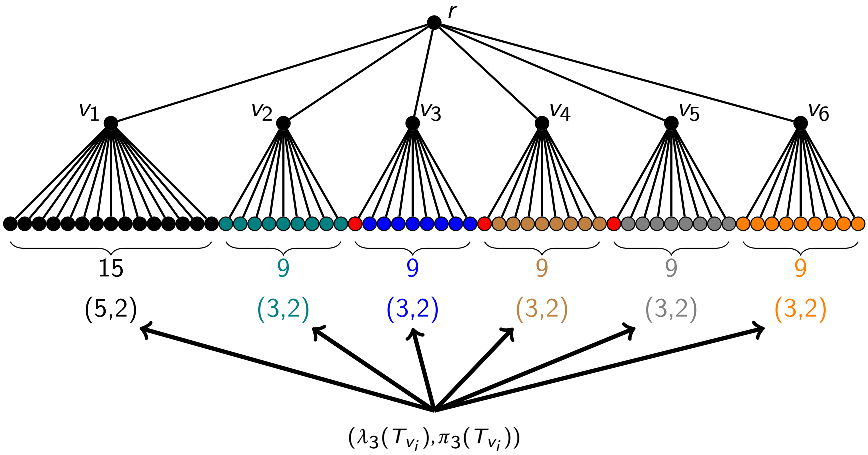

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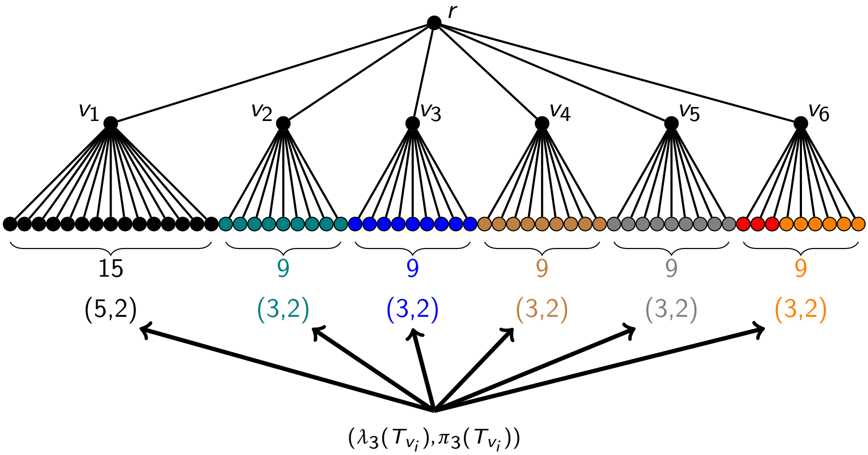

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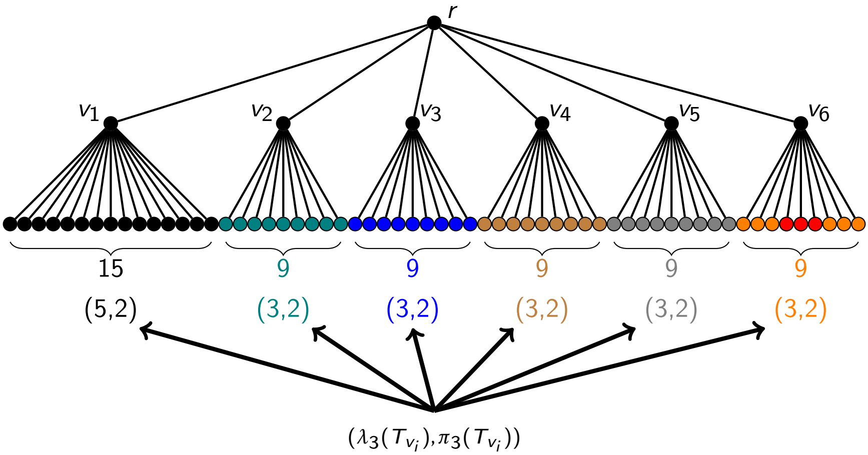

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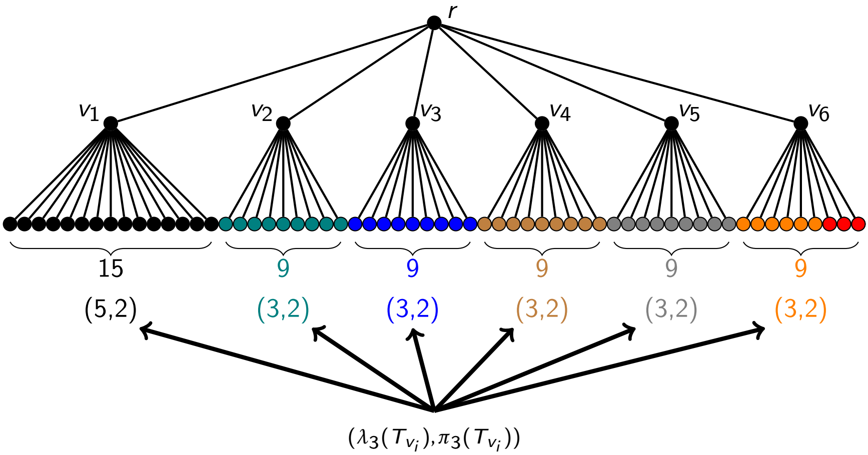

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Also:

- For each pair $(\lambda_k(T_v), \pi_k(T_v))$, can retrieve corresponding strategies.
- $\binom{n}{k}$ possible first steps; polynomial when k is a constant.

Conclusion and perspectives

Future work and questions

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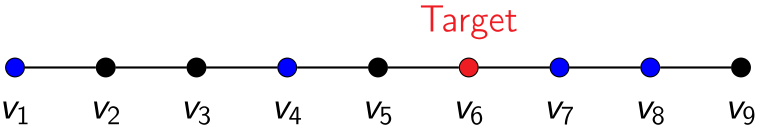
Thank you for your attention!

Locating via relative distances

From probed vertices, get relative distances to the target instead.



Each $v_i \rightarrow$ Vector of relative distances to the probed vertices:

$$v_1 \rightarrow (\{v_1\},\{v_4\},\{v_7\},\{v_8\}) \quad v_2 \rightarrow (\{v_1\},\{v_4\},\{v_7\},\{v_8\}) \quad v_3 \rightarrow (\{v_4\},\{v_1\},\{v_7\},\{v_8\})$$

$$v_4 \rightarrow (\{v_4\},\{v_1,v_7\},\{v_8\}) \quad v_5 \rightarrow (\{v_5\},\{v_7\},\{v_8\},\{v_1\}) \quad v_6 \rightarrow (\{v_7\},\{v_4,v_8\},\{v_1\})$$

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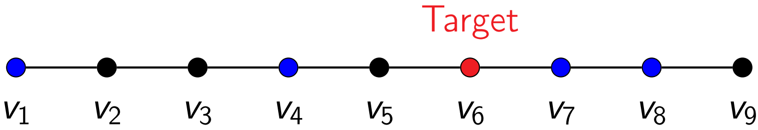

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Arising notions of:

- Centroidal set (Foucaud, Klasing, Slater, 2014);
- Centroidal dimension (Foucaud, Klasing, Slater, 2014);
- Sequential centroidal dimension (us, 2018+).

Decision problems related to the last notion are NP-complete...