Strong edge-coloring of $(3, \Delta)$-bipartite graphs

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LIRMM
March 12th, 2015

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G: undirected simple graph
c: edge-coloring of G

Definition: *strong edge-coloring*

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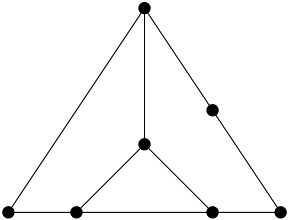

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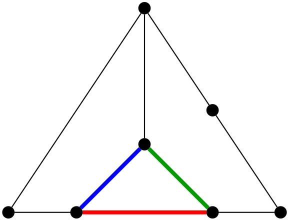

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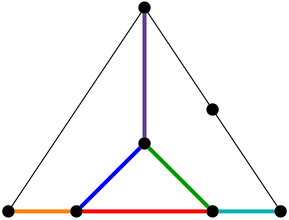

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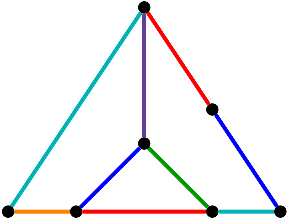

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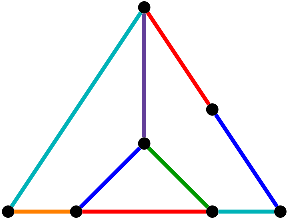

Equivalently:

- edge-partition giving *induced* matchings
- proper vertex-coloring of $L(G)^2$

Strong chromatic index

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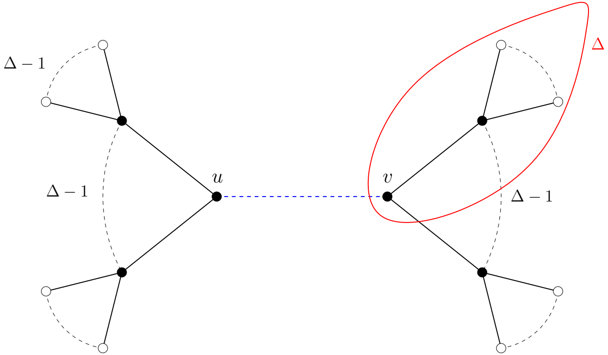
Brooks-like argument: $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ ($\approx 2\Delta^2$)



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optimality of $2\Delta^2$?

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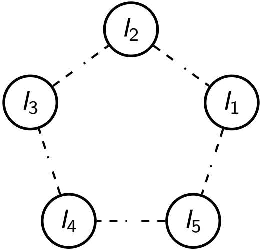

- every I_j is an independent set,
- two "adjacent" I_j's are complete to each other,
- if $\Delta = 2k$, then $|I_j| = k$,
- if $\Delta = 2k+1$, then $|I_1| = |I_2| = |I_3| = k$,

  and $|I_4| = |I_5| = k+1$.

Erdős and Nešetřil's conjecture

Conjecture [Erdős, Nešetřil – 1985]

We have $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 \text{ for } \Delta \text{ even, and} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \text{ otherwise.} \end{cases}$

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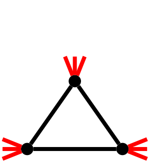
- for $\Delta = 4$, we know $\chi'_s(G) \leq 22$ [Cranston – 2006]

Beyond Erdős and Nešetřil's construction

Less dependencies for graphs with no small cycles



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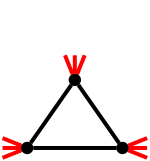 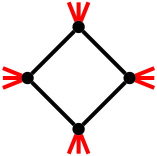 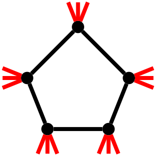

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If G is C_4-free, then $\chi'_s(G) \leq (2 + o(1))\frac{\Delta^2}{\ln \Delta}$.

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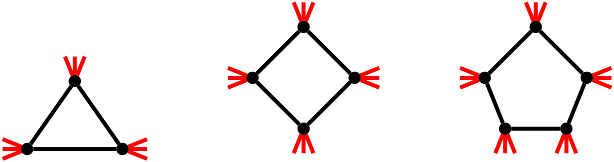

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What for C_3- and C_5-free graphs?

What for bipartite graphs?

Bipartite graphs are C_3 and C_5-free...

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If G is bipartite, then $\chi'_s(G) \leq \Delta^2$.

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$G = (A, B, E)$: bipartite graph with bipartition A and B
(Δ_A, Δ_B)-bipartite graph: A and B have maximum degree Δ_A and Δ_B, resp.

Conjecture [Brualdi, Quinn Massey – 1993]

If G is (Δ_A, Δ_B)-bipartite, then $\chi'_s(G) \leq \Delta_A \Delta_B$.

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We confirm the conjecture when $\Delta_A = 3$ and $\Delta_B \geq 4$

Theorem [B., Lagoutte, Valicov – 2014+]

If G is $(3, \Delta_B)$-bipartite, then $\chi'_s(G) \leq 4\Delta_B$.

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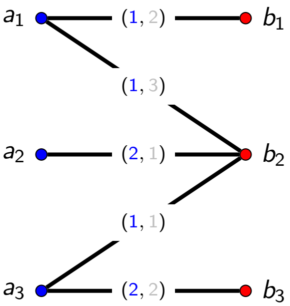

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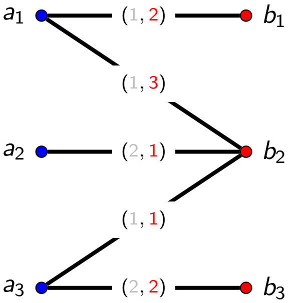

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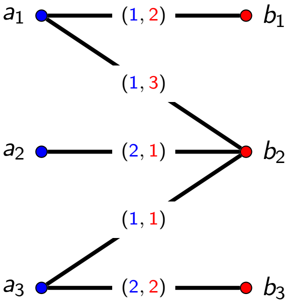

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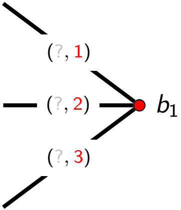
As c_B, just consider a proper Δ_B-incidence coloring



(adjacent incidences are of the form (b_1, e) and (b_1, f))

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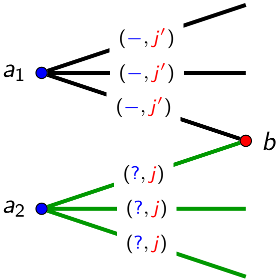

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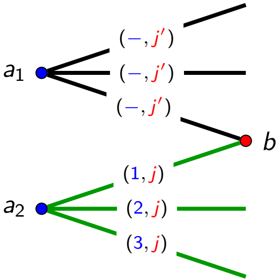

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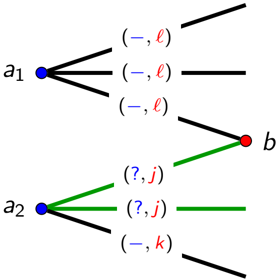

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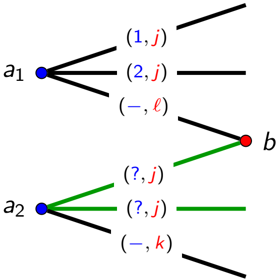

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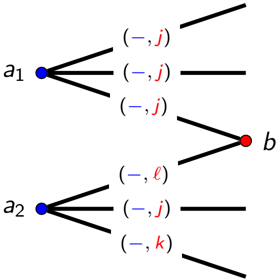

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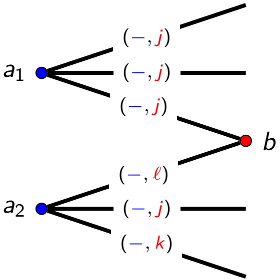

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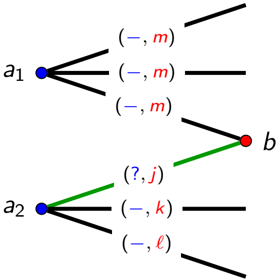

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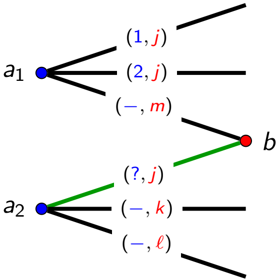

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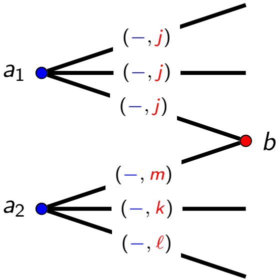

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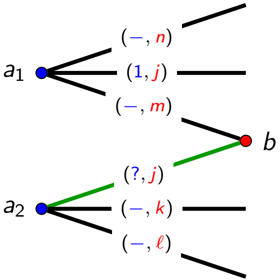

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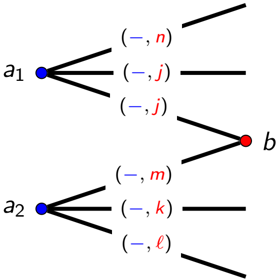

... we could switch two colors and make a_1 Type 2

Step 3: color the edges incident to Type 3 vertices

Just assign the colors among $\{1, 2, 3\}$ greedily

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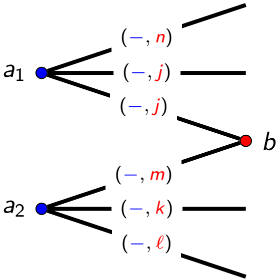

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Step 4: color the lonely edges incident to Type 2 vertices

$\mathcal{C}_j$: (connected) subgraph induced by the j-lonely edges (*i.e.* with $c_B = j$)

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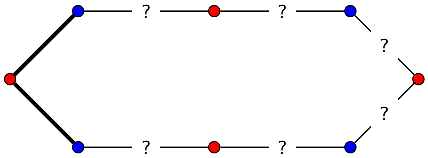

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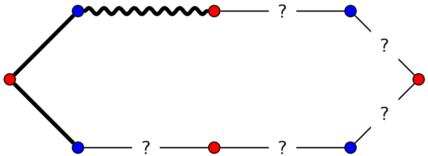

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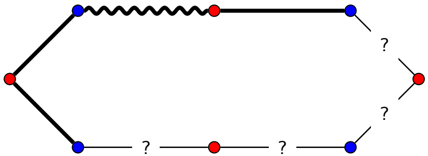

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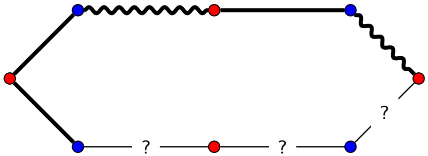

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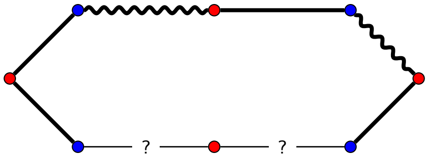

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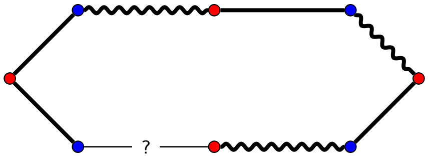

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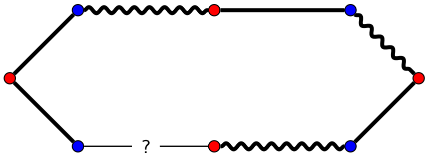

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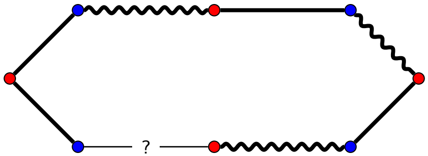

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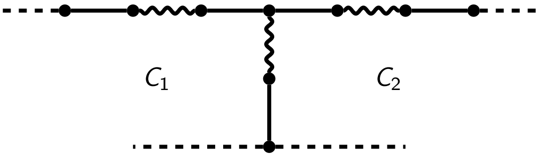

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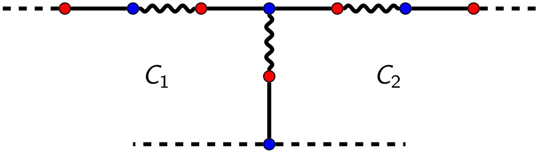

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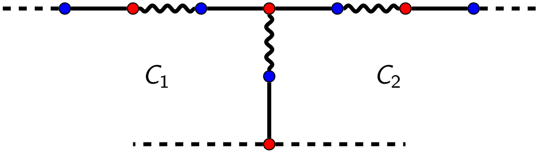

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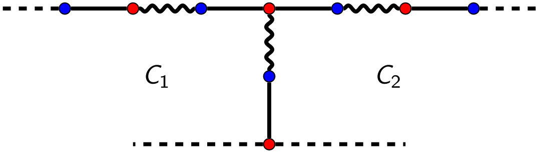

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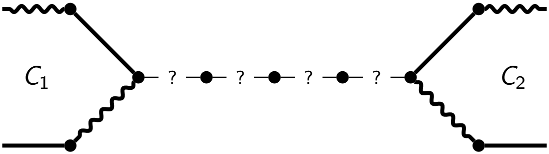

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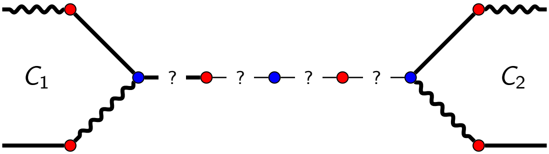

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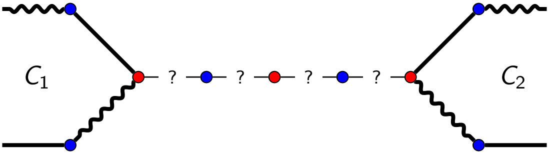

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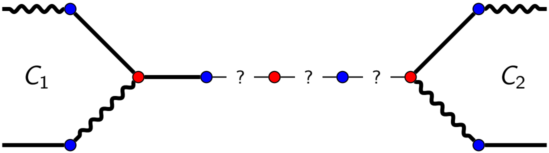

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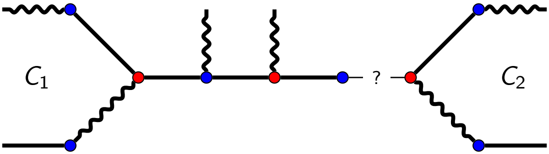

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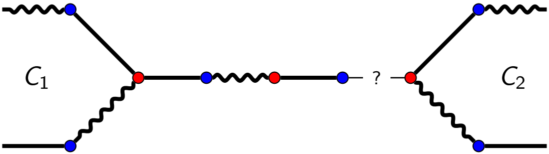

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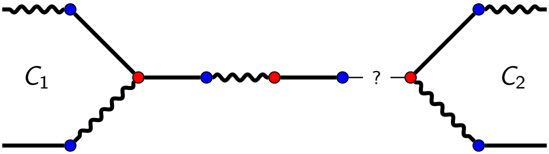

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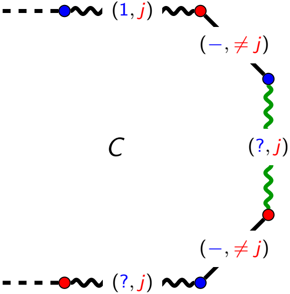

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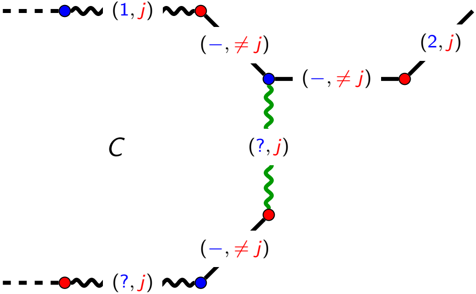

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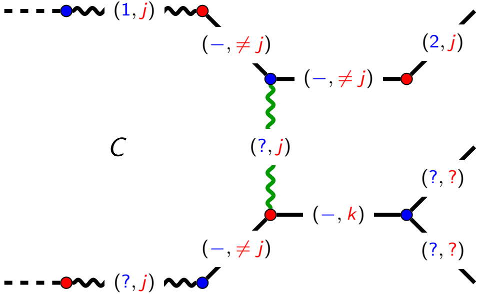

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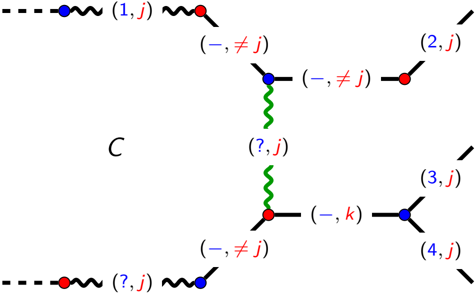

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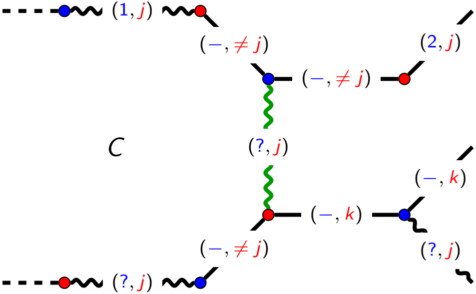

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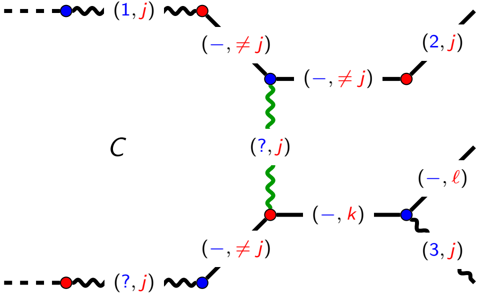

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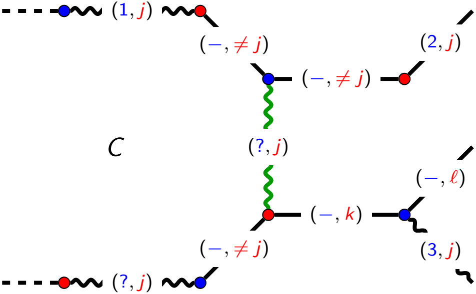

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Step 4 – the end!

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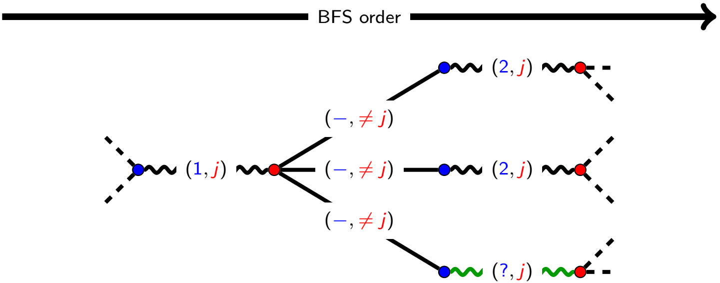

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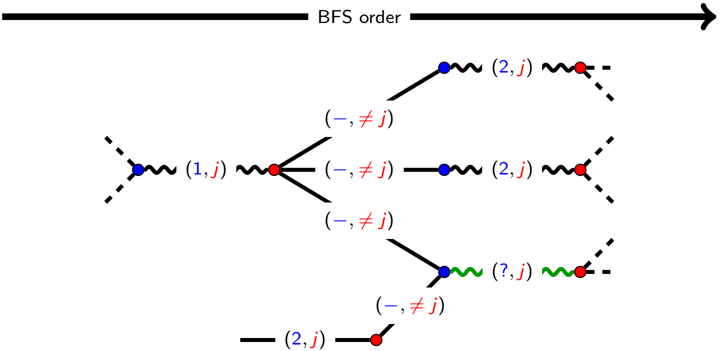

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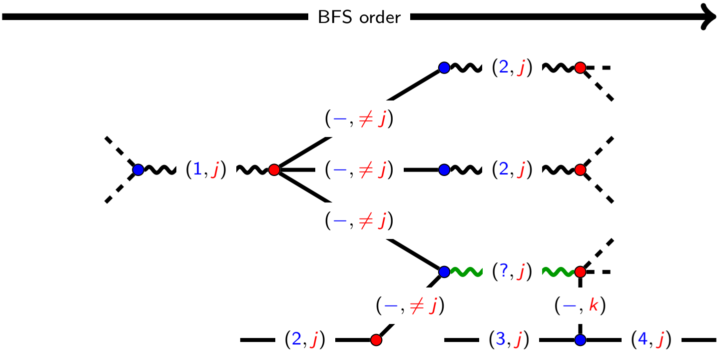

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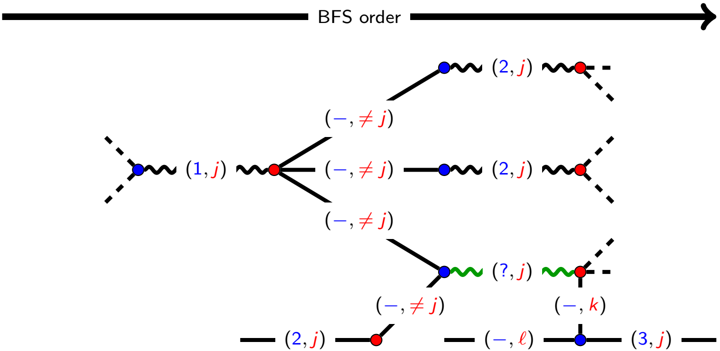

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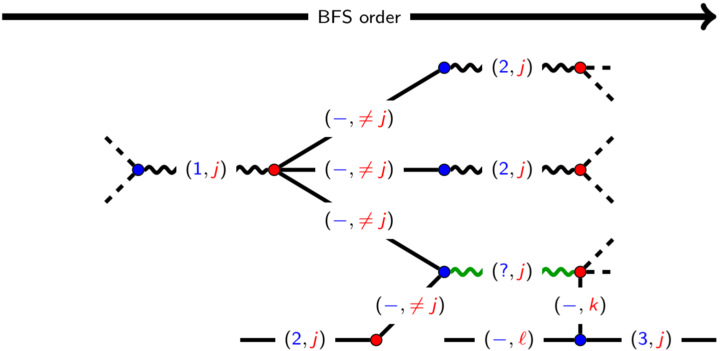

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