# – The 1-2-3 Conjecture –
# – 10 years after –

## I) Motivations and origins

$G$ = simple undirected graph.

$G$ regular = all vertices have the same degree

"irregularity"? (antonym notion)

$G$ totally irregular = all vertices have distinct degrees

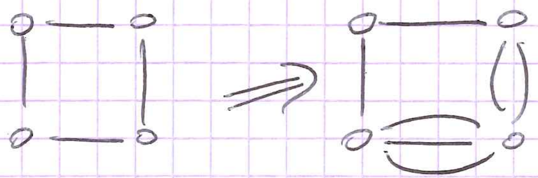
Pb: only $K_1$ is totally irregular!

Proof: in a totally irregular simple graph, with order $n \geq 2$, the degree sequence should be $(0, 1, \ldots, n-1)$, which is impossible. ■

× Solution 1 [Chartrand, Jacobson, Lehel, Oellermann, Ruiz, Saba - 1988]:

"multiply" the edges of a simple graph to get a totally irregular multigraph.

Example:





Rk: edge-multiplications preserve the adjacencies.

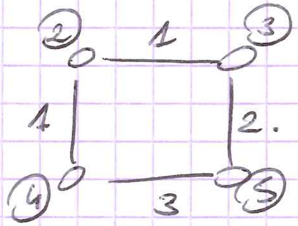
Pb: Doing this transformation with minimizing the maximum # of times an edge is multiplied?

Nt: For $G$, this parameter is denoted by $s(G)$, the irregularity strength of $G$.

Example: $s(C_4) = 3$

Rk: $s(K_2) = \infty$ so we consider graphs with no isolated $K_2$'s.

Rk2: Can be seen as a weighting problem: give weights among $\{1, 2, \ldots, k\}$ to edges so that the sums of incident weights at each vertex yield an injective vertex-colouring. So $s(G)$ is the least $k$ s.t. such edge-weightings exist.




Theorem [Nierhoff - 2000]:

$\forall G, \; s(G) \leq |V(G)| - 1$.

Reached e.g. for stars. Still some investigations on $s(G)$, even for trees: what is the exact value for a given tree?

$n_i$ : # of vertices w/ degree $i$.

$|\{1,2,\dots,k\}| \geq n_1$

$|\{2,3,\dots,2k\}| \geq n_2$

...

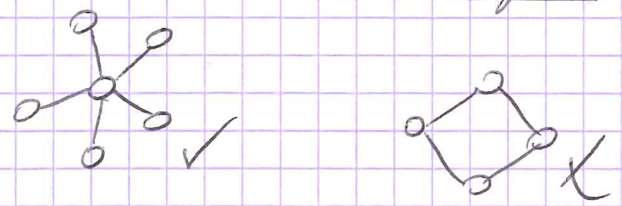
For trees, exact values should involve $n_1$ and $n_2$ only.

x Solution 2 [Chartrand, Erdős, Oellermann - 1988]

"weaken" the notion of irregularity.

G locally irregular = all adjacent vertices have distinct degrees.

Examples:

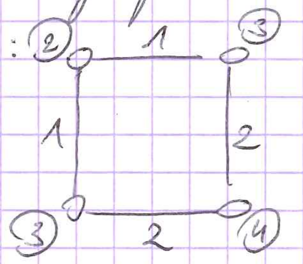



Again, if a simple graph is not locally irregular, then we would like to multiply its edges to get a locally irregular multigraph.



Nt: least # of needed consecutive edge weights / max # edge mul. is denoted $\chi^e_{\Sigma}(G)$ for a graph G.

Example:




$\chi^e_{\Sigma}(C_4)=2$

Rk: Again $\chi^e_{\Sigma}(K_2)=\infty$

1-2-3 Conjecture [Karoński, Łuczak, Thomason - 2004]:

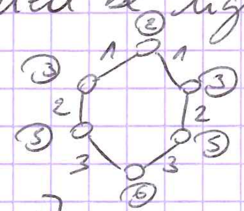
$\forall G \quad \chi^e_{\Sigma}(G) \leq 3$.

In other words, we should be able to "encode" a proper vertex-colouring of any graph using at most three edge weights.

II) The 1-2-3 Conjecture.

Rk: if true, the "3" would be tight.

E.g. $\chi^e_{\Sigma}(C_6)=3$



But...

Theorem [Dudek, Wajc - 2011]:

Deciding whether $\chi^e_{\Sigma}(G) \leq 2$ is NPC.

However ...

**Theorem [Addario-Berry, Dalal, Reed - 2008].**
Let $G$ be a random graph from $G_{n,p}$.
Then a.a.s. $\chi^e_\Sigma(G) \le 2$.

Towards the 1-2-3 Conjecture ...

- $\chi^e_\Sigma(G) \le 183$ [Karoński, Łuczak, Thomason - 2004]
- $\chi^e_\Sigma(G) \le 30$ [A-B, Dalal, McDiarmid, Reed, Thomason - 2007]
- $\chi^e_\Sigma(G) \le 16$ [A-B, Dalal, Reed - 2008]
- $\chi^e_\Sigma(G) \le 13$ [Wang, Yu - 2008]

Finally ...

**Theorem [Kalkowski, Karoński, Pfender - 2010]:**
$\forall G$, $\chi^e_\Sigma(G) \le 5$.

Proof: see 1-2 Conjecture ... ■

The 1-2-3 Conjecture is verified for many families of graphs such as bipartite graphs, 3-colourable graphs, some planar graphs (discharging method).
Except the conjecture itself, the major open question is:



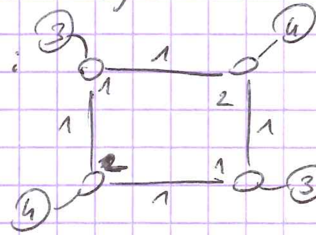
Question: Is there an "easy" characterization of bipartite graphs w/ $\chi^e_\Sigma \le 2$?

Many partial results but still open ... In particular, is the NP-Completeness result true for bipartite graphs?

## III) The 1-2 Conjecture.

Instead of weighting the edges only, also a "local weight" at each vertex (~ there is a loop at each vertex).

Nt. $\chi^t_\Sigma(G)$: least # of weights in a total-weighting of $G$ yielding a proper vertex-colouring (by sums).

Example: 


$\chi^t_\Sigma(C_4) = 2$.

Rk: $\chi^t_\Sigma(G) \le \chi^e_\Sigma(G)$ $\forall G$ and
$\chi^t_\Sigma$ defined for all graphs.

Rk: Maybe the other way around also...

Conjecture [Przybyło, Woźniak - 2010]:

$\forall G$ , $\chi^t_{\Sigma}(G) \leq 2$.

Again the 1-2 Conjecture is true for many graph classes (complete, 3-colourable, 4-regular, some planar, etc...).

Major result:

theorem [Kalkowski - 2008]:

$\forall G$ , $\chi^t_{\Sigma}(G) \leq 3$.

Proof: Process the vertices following an arbitrary ordering. Start w/ all edges weighted 2 and vertices weighted 1. Each time a vertex is considered, we "fix" its "final sum" $f$ and define its "current sum" $c$ as $f = c$. At every step, either $f = c$ or $c = f - 1$ for each vertex. When processing $v$, $f$ is chosen in such a way that no problem w/ the previously considered $f$'s. If $k$ backward neighbours, at least $k+1$ possible values. At the end, we change the vertex weight of any vertex if $c = f - 1$. ■



Many consequences...

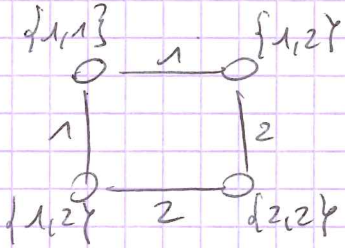
- → "1-2-3-4-5 theorem".
- → New bounds on the irregularity strength.

## IV) Derived versions.

We can change the distinguishing aggregate, weight the edges and/or the vertices, change the distance between the vertices to distinguish.

### A) Multiset versions.

Nt: $\chi^e_m(G)$: least # of edge colours s.t. we may distinguish the adjacent vertices by their multisets of incident colours.

Example: 


$\chi^e_m(C_4) = 2$.

Conjecture [A-B, Aldred, Dalal, Reed - 2005].

$\chi^e_m(G) \leq 3$ $\forall G$.

Rk: Again, would be tight (consider $C_6$) and the decision problem is NPC for 2.

Rk: $\chi^e_{\Sigma}(G) \le k \Rightarrow \chi^e_m(G) \le k$

so $\chi^e_m(G) \le 5 \quad \forall G$.

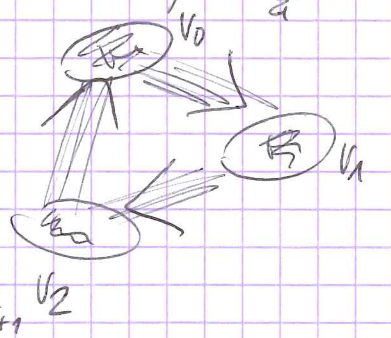
But ...

Theorem [A-B, Aldred, Dalal, Reed - 2005]

$\chi^e_m(G) \le 4 \quad \forall G$.

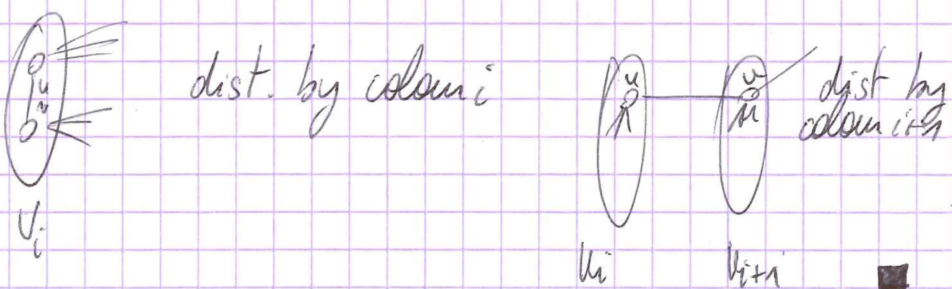
Proof: easy if $\chi(G) \le 3$ (for $\chi^e_{\Sigma}(G) \le 3$ then)

Otherwise a partition




s.t. $\begin{cases} \bullet\ \forall v \in V_i, \text{ more neighb. in } V_{i+1} \\ \bullet\ \forall v \in V_i, \text{ an edge to } V_{i+1} \end{cases}$

Colour $i$ all edges of $V_i$ + edges to $V_{i+1}$ to distinguish "inside" $V_i$. All other edges are given another same 4th colour.






Again the bipartite case is unclear ...

But ...

Theorem [Havet, Paramaguru, Sampathkumar - 2012]

$\forall G$ bipartite, $\chi^e_m(G) \le 2$ if $\delta(G) \ge 3$.

B) Product versions.

Nt: Same as previously but for products of incident weights $\Rightarrow$ $\chi^e_{\Pi}(G)$ and $\chi^t_{\Pi}(G)$.

Rk: weight 1 does nothing.

Conjectures [Skowronek-Kaziów - 2012]:

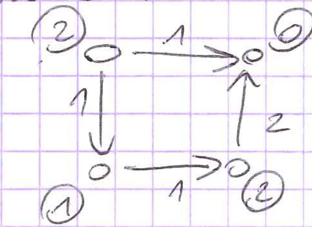
$\chi^e_{\Pi}(G) \le 3$ & $\chi^t_{\Pi}(G) \le 2$

She proved $\chi^e_{\Pi}(G) \le 4$ and $\chi^t_{\Pi}(G) \le 3$ using mainly the tripartition $\sqsubset$.

C) Outsums for oriented graphs.

Def: $\chi^e_{\Sigma^+}(\vec{G})$ = least # of weights so that adjacent vertices of $\vec{G}$ get distinguished by their "outsums".

Example:

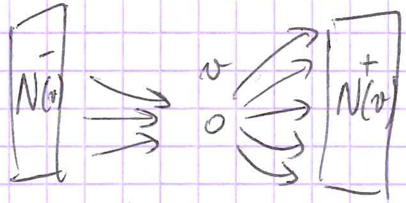



$\chi^e_{\Sigma^+}(\vec{C}_4) = 2$.

theorem [Baudon, B., Sopena - 2015]. (M')

$\forall \vec{G} \quad \chi^{e}_{\Sigma^{1,+}}(\vec{G}) \le 3.$

Proof: Choose $v$ with $d^+(v) \geq d^-(v)$.




use induction.

$2d^+(v)+1$ outsums for $v$, but only $\leq 2d^+(v)$ neighbours. Weighting does not alter the other outsums. ■

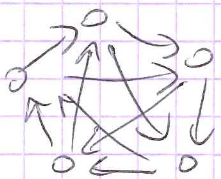
Rk: Arguments correct for list & product vers.

Graphs w/ $\chi^{e}_{\Sigma^{1,+}} \le 2$?

theorem [Baudon, B., Sopena - 2015].

No "easy" characterization exists, unless P=NP.

Rk: The 1-2 Conjecture analogue is wrong:



2 outregular $K_5$.

outsums among $\{3,4,5,6\}$ ↯.