On the hardness of determining the irregularity strength of graphs

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General context

From regularity to irregularity

regular graph ☺

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thus n possible degrees
all distinct and $n \geq 2 \Rightarrow$ **isolated vertex + universal vertex** $\lightning$

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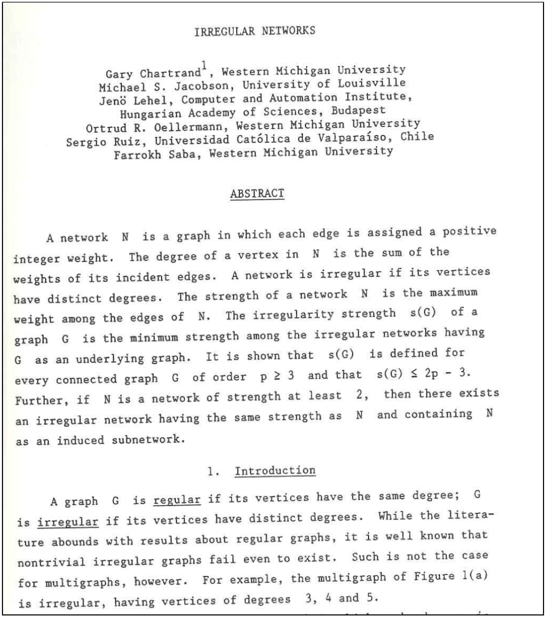
How to overcome this?



IRREGULAR NETWORKS

Gary Chartrand[1], Western Michigan University
Michael S. Jacobson, University of Louisville
Jenö Lehel, Computer and Automation Institute, Hungarian Academy of Sciences, Budapest
Ortrud R. Oellermann, Western Michigan University
Sergio Ruiz, Universidad Católica de Valparaíso, Chile
Farrokh Saba, Western Michigan University

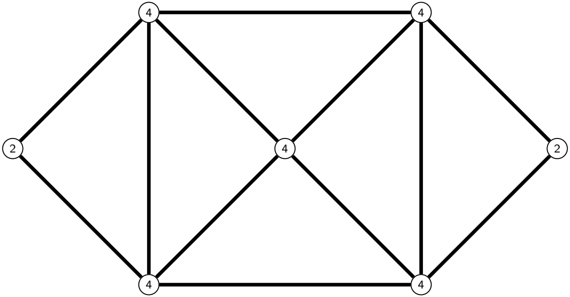
ABSTRACT

A network N is a graph in which each edge is assigned a positive integer weight. The degree of a vertex in N is the sum of the weights of its incident edges. A network is irregular if its vertices have distinct degrees. The strength of a network N is the maximum weight among the edges of N. The irregularity strength $s(G)$ of a graph G is the minimum strength among the irregular networks having G as an underlying graph. It is shown that $s(G)$ is defined for every connected graph G of order $p \geq 3$ and that $s(G) \leq 2p - 3$. Further, if N is a network of strength at least 2, then there exists an irregular network having the same strength as N and containing N as an induced subnetwork.

1. Introduction

A graph G is regular if its vertices have the same degree; G is irregular if its vertices have distinct degrees. While the literature abounds with results about regular graphs, it is well known that nontrivial irregular graphs fail even to exist. Such is not the case for multigraphs, however. For example, the multigraph of Figure 1(a) is irregular, having vertices of degrees 3, 4 and 5.

Sample example



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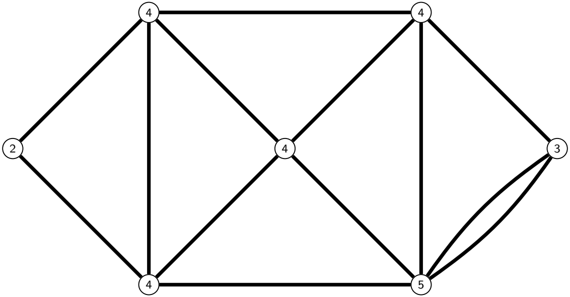

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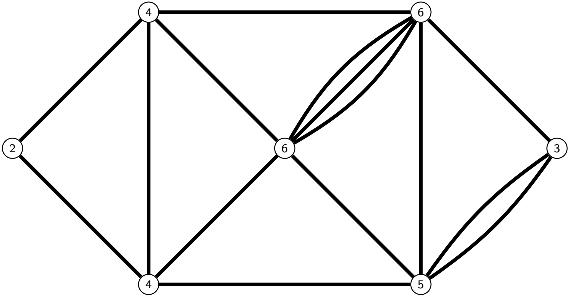

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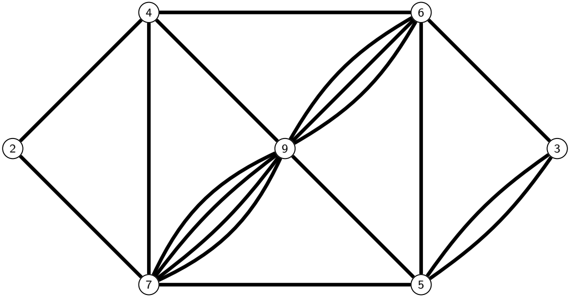

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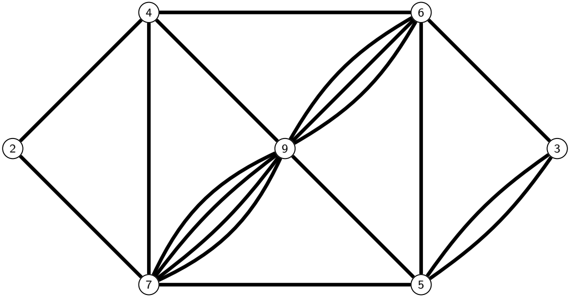

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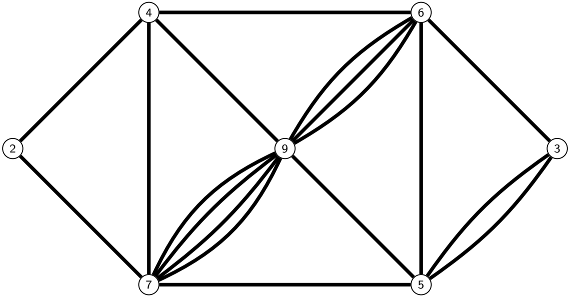

- graph $\rightarrow$ (irregular) multigraph
- preserves the **original structure**
- Chartrand *et al.*: avoid "exploding" an edge too much?
- above: every edge $\rightarrow \leq 4$ parallel edges; what about ≤ 3?

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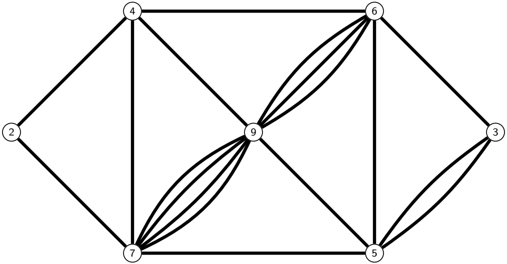

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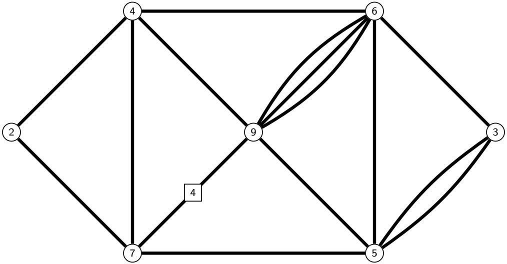

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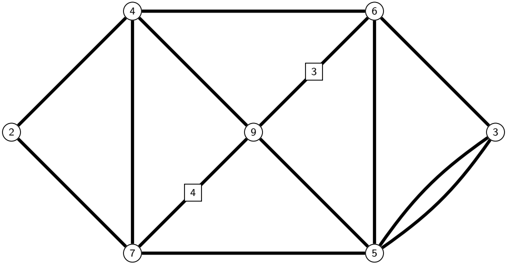

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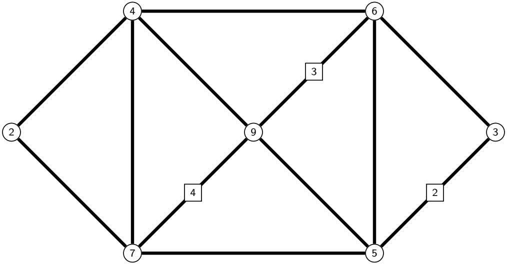

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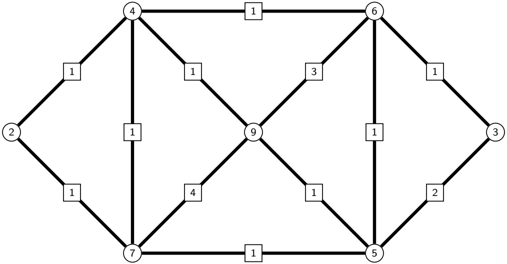

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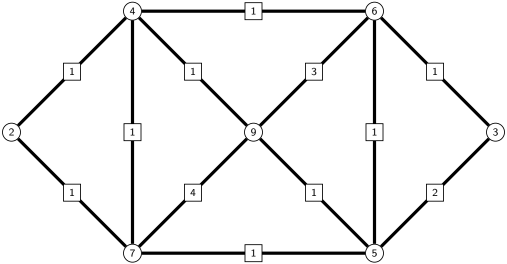

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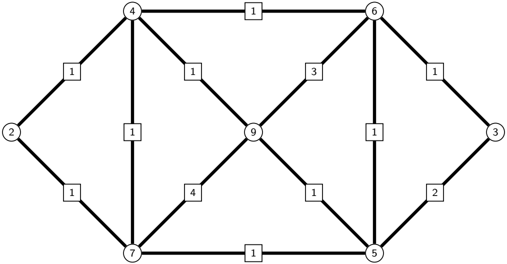

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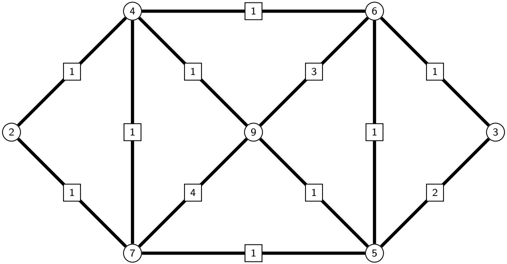

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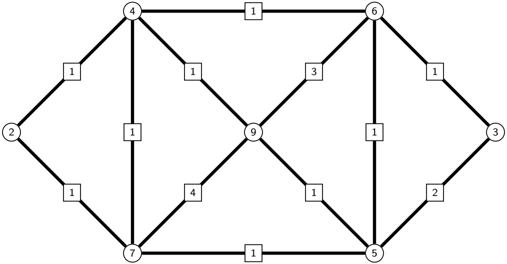

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- degrees $\rightarrow$ incident sums
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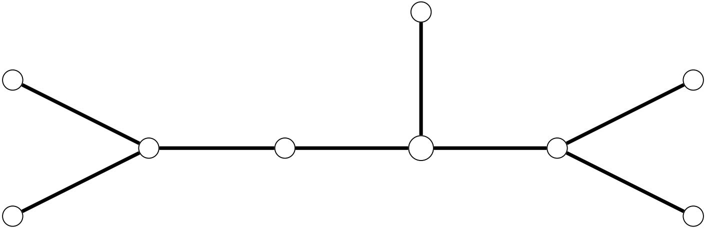
- minimising max. edge "explosion" $\rightarrow$ minimising max. label
- *irregularity strength $s(G)$* of G: this minimum

A few more examples



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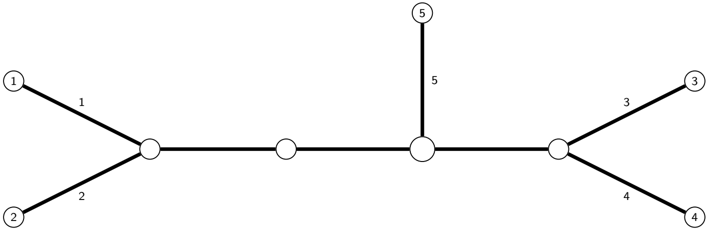

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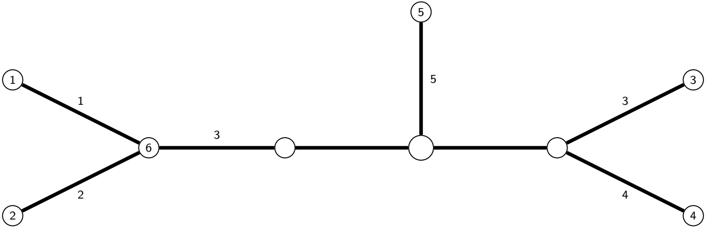

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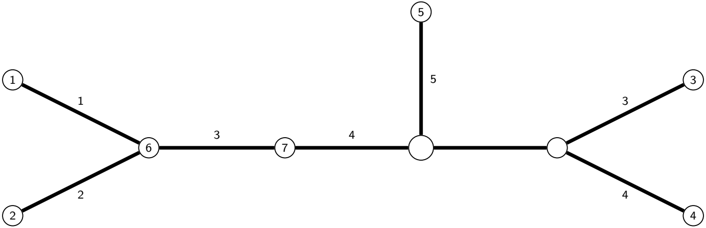

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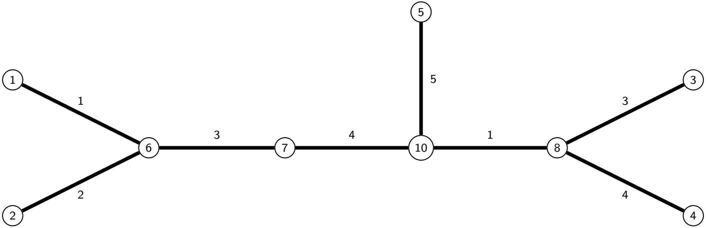

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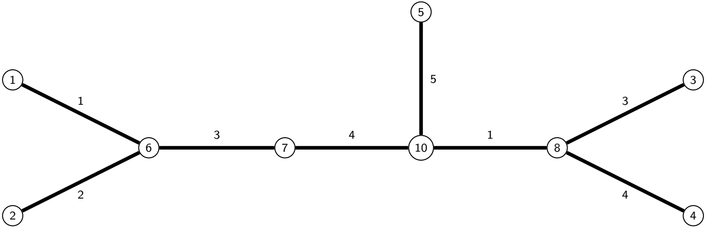
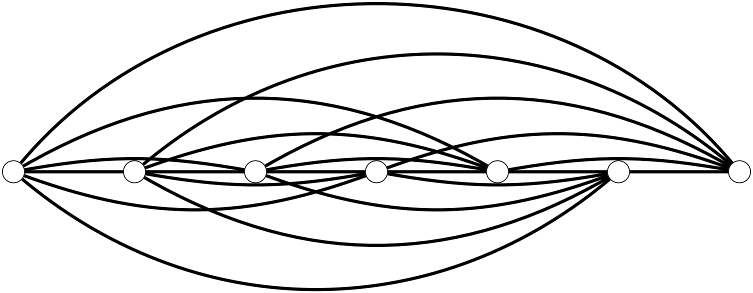


$$s(G) = 5$$

A few more examples



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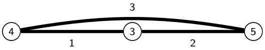

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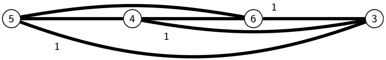

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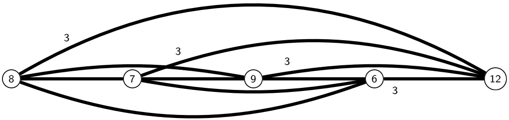

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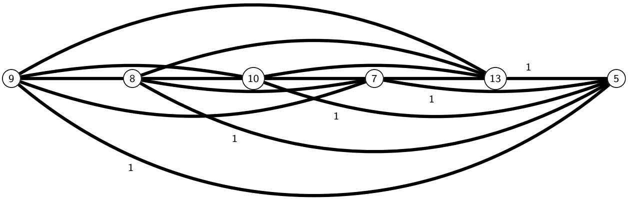

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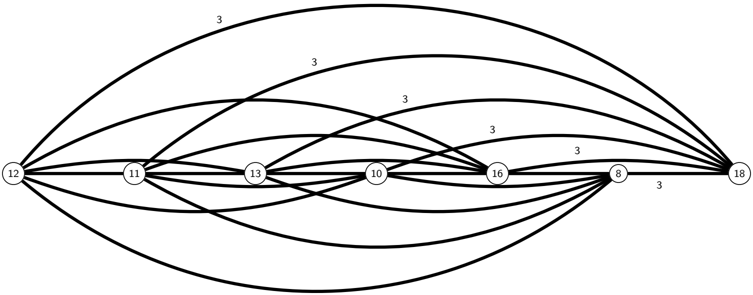

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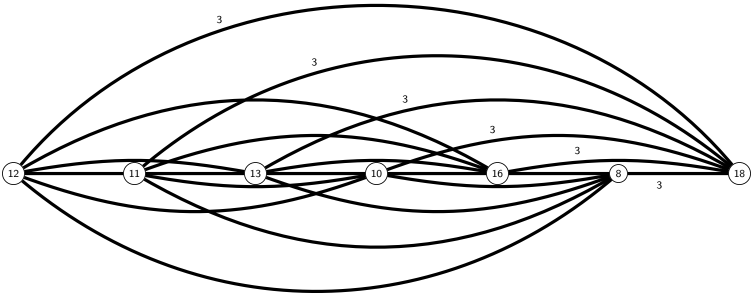

$s(K_7) \leq 3$ (and actually $s(K_7) = 3$)

Understanding the problem

Remarks:

- $s(G)$ well defined iff G is *nice* (no K_2 as a connected component)

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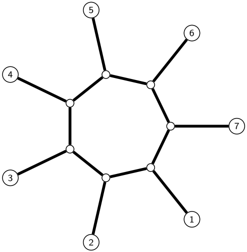
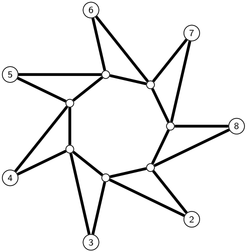
for any $x \geq 0$, set $\mathrm{nb}(x)$ as the # of degree-x vertices; then, need:

- $\mathrm{nb}(1) \leq k$ — for $x = 1$, sums in $\{1, \dots, k\}$
- $\mathrm{nb}(2) \leq 2k - 1$ — for $x = 2$, sums in $\{2, \dots, 2k\}$
- $\mathrm{nb}(3) \leq 3k - 2$ — for $x = 3$, sums in $\{3, \dots, 3k\}$
- etc.





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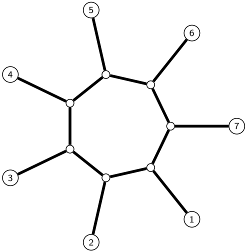

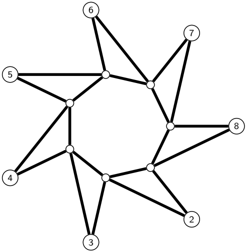

but vertices with different, yet close degrees can also “collide” ☹

Some known results

lots (lots!) of results of varying interest...

7.14 Irregular Total Labelings

Motivated by the notion of the irregularity strength of a graph introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz, and Saba [309] in 1988 and various kinds of other total labelings, Bača, Jendroľ, Miller, and Ryan [136] introduced the *total edge irregularity strength* of a graph as follows. For a graph $G(V, E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an *edge irregular total k-labeling* if for every pair of distinct edges uv and xy, $\partial(u)+\partial(uv)+\partial(v) \neq \partial(x)+\partial(xy)+\partial(y)$. Similarly, ∂ is called an *vertex irregular total k-labeling* if for every pair of distinct vertices u and v, $\partial(u) + \sum \partial(e)$ over all edges e incident to $u \neq \partial(v) + \sum \partial(e)$ over all edges e incident to v. The minimum k for which G has an edge (vertex) irregular total k-labeling is called the *total edge (vertex) irregularity strength* of G. The total edge (vertex) irregular strength of G is denoted by tes(G) (tvs(G)). They prove: for $G(V, E)$, E not empty, $\lceil(|E| + 2)/3\rceil \leqslant \text{tes}(G) \leqslant |E|$; $\text{tes}(G) \geqslant \lceil(\Delta(G)+1)/2\rceil$ and $\text{tes}(G) \leqslant |E| - \Delta(G)$, if $\Delta(G) \leqslant (|E|-1)/2$; $\text{tes}(P_n) = \text{tes}(C_n) = \lceil(n+2)/3\rceil$; $\text{tes}(W_n) = \lceil(2n + 2)/3\rceil$; $\text{tes}(C_3^n)$ (friendship graph) $= \lceil(3n + 2)/3\rceil$; $\text{tvs}(C_n) = \lceil(n + 2)/3\rceil$; for $n \geqslant 2$, $\text{tvs}(K_n) = 2$; $\text{tvs}(K_{1,n}) = \lceil(n + 1)/2\rceil$; and $\text{tvs}(C_n \times P_2) = \lceil(2n + 3)/4\rceil$. Jendroľ, Miškul, and Soták [610] (see also [611]) proved: $\text{tes}(K_5) = 5$; for $n \geqslant 6$, $\text{tes}(K_n) = \lceil(n^2 - n + 4)/6\rceil$; and that $\text{tes}(K_{m,n}) = \lceil(mn + 2)/3\rceil$. They conjecture that for any graph G other than K_5, $\text{tes}(G) = \max\{\lceil(\Delta(G) + 1)/2\rceil, \lceil(|E| + 2)/3\rceil\}$. Ivančo and Jendroľ [601] proved that this conjecture is true for all trees. Jendroľ, Miškuf, and Soták [610] prove the conjecture for complete graphs and complete bipartite graphs. Ahmad and Bača [46] proved the conjecture holds for the categorical product of two paths. (The categorical product $P_m \times P_n$ has vertex set the Cartesian product of P_m and P_n and edge set $((u, x), (v, y))$ for all (u, v) in P_m and (x, y) in P_n.) Brandt, Misškuf, and Rautenbach [260] proved the conjecture for large graphs whose maximum degree is not too large relative to its order and size. In particular, using the probabilistic method they prove that if $G(V, E)$ is a multigraph without loops and with nonzero maximum degree less than $|E|/10^3\sqrt{8|V|}$, then $\text{tes}(G) = (\lceil|E| + 2)/3\rceil$. As corollaries they have: if $G(V, E)$ satisfies $|E| \geqslant 3 \cdot 10^3|V|^{3/2}$, then $\text{tes}(G) = \lceil(|E| + 2)/3\rceil$; if $G(V, E)$ has minimum degree $\delta > 0$ and maximum degree Δ such that $\Delta < \delta\sqrt{|V|}/10^3 \cdot 4\sqrt{2}$ then $\text{tes}(G) = \lceil(|E| + 2)/3\rceil$; and for every positive integer Δ there is some $n(\Delta)$ such that every graph $G(V, E)$ without isolated vertices with $|V| \geqslant n(\Delta)$ and maximum degree at most Δ satisfies $\text{tes}(G) = \lceil(|E| + 2)/3\rceil$. Notice that this last result includes d-regular graphs of large order. They also prove that if $G(V, E)$ has maximum degree $\Delta \geqslant 2|E|/3$, then G has an edge irregular total k-labeling with $k = \lceil(\Delta + 1)/2\rceil$. Pfender [984] proved the conjecture for graphs with at least 7×10^{10} edges and proved for graphs $G(V, E)$ with $\Delta(G) \leqslant E(G)/4350$ we have $\text{tes}(G) = (\lceil|E| + 2)/3\rceil$.

Nurdin, Baskoro, Salman, and Gaos [964] determine the total vertex irregularity strength of trees with no vertices of degree 2 or 3; improve some of the bounds given in [136]; and show

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- variants (local, total, etc.)

Our (modest ☺) contribution

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- obvious for $k = 1$ ☺ . so, what about

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- get a better grasp on these labellings

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- show you most of the two proofs
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Our results

Main question for today

For a given graph G, determining whether $s(G) \leq 2$?

towards this:

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Let's go ☺ ☺ ☺ !!

Distant irregularity strength

(Very brief) introduction

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- regarding our question, for $d = 1$:

Theorem [Dudek, Wajc, 2011]

For a given graph G, determining whether $s^1(G) \leq 2$ is NP-complete.

Our result

Theorem [B., 2022]

For any $d \geq 1$, and any given graph G,
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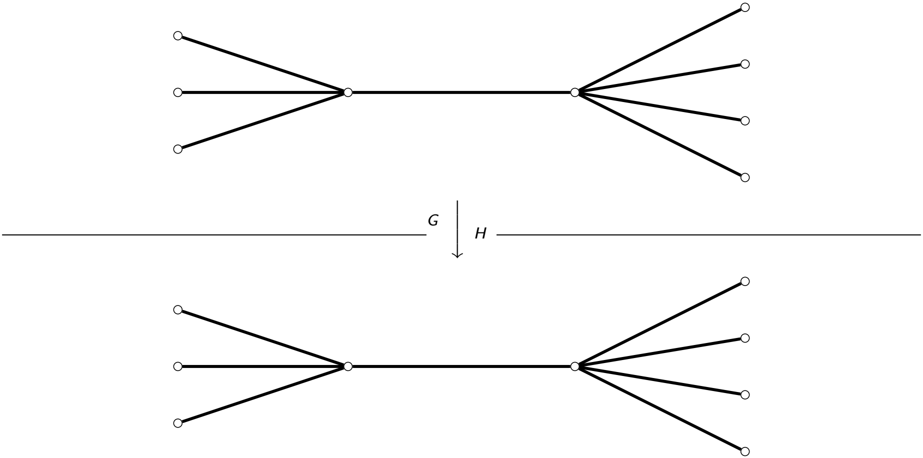

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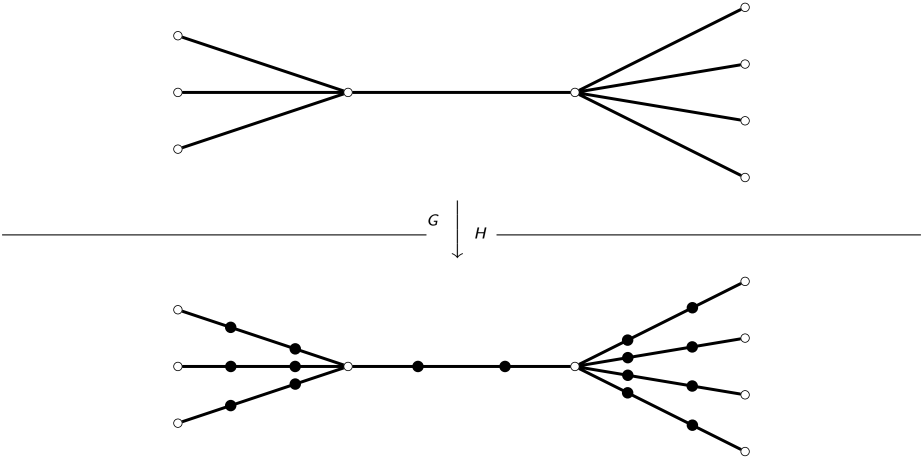

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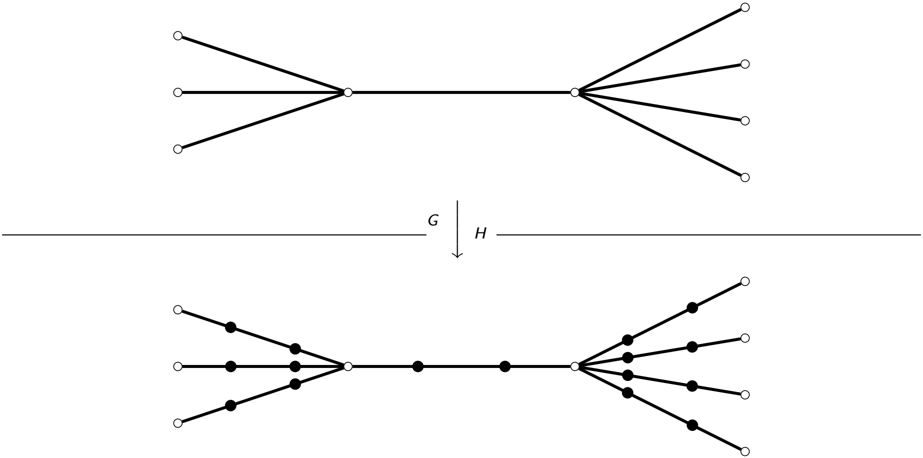

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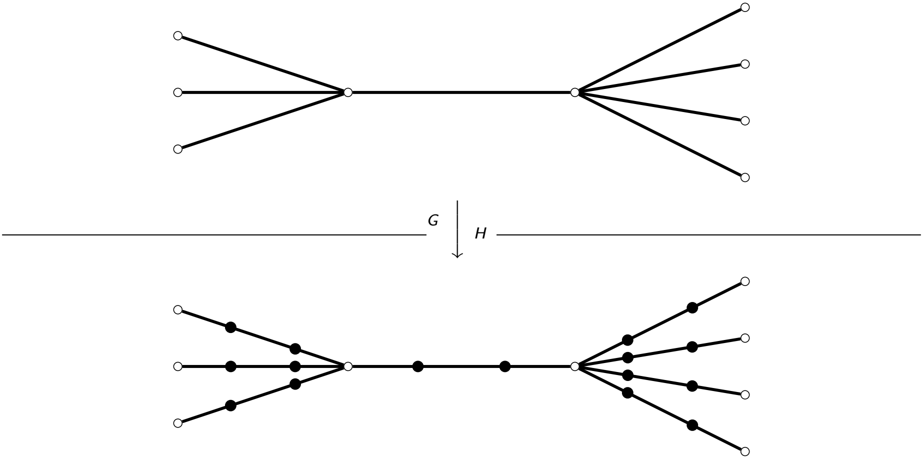

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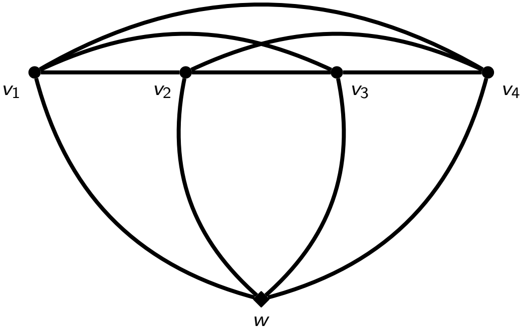
however:

Lemma [B., 2022]

Assume the vertices of K_{p+1} are $w, v_1, \ldots, v_p$. By every 2-labelling that is distinguishing when omitting w, the set $\{\sigma(v_1), \ldots, \sigma(v_p)\}$ is either $\{p, \ldots, 2p-1\}$ or $\{p+1, \ldots, 2p\}$. Furthermore, for every $s \in \{p, 2p\}$, there exist distinguishing 2-labellings of K_{p+1} where $s \notin \{\sigma(w), \sigma(v_1), \ldots, \sigma(v_p)\}$, and $\sigma(w)$ is either $\frac{3p}{2}$ (even p), or (odd p) $\frac{3p-1}{2}$ ($s = 2p$) or $\frac{3p+1}{2}$ ($s = p$).

Illustration



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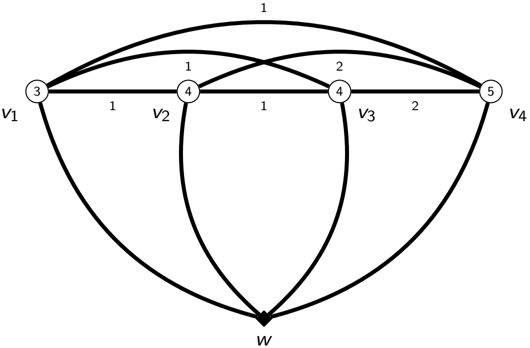

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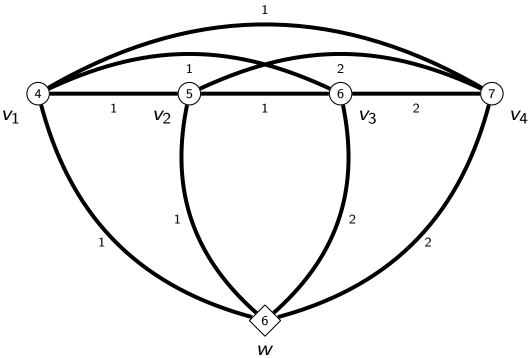

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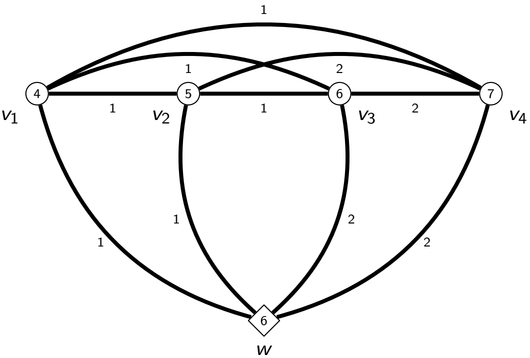

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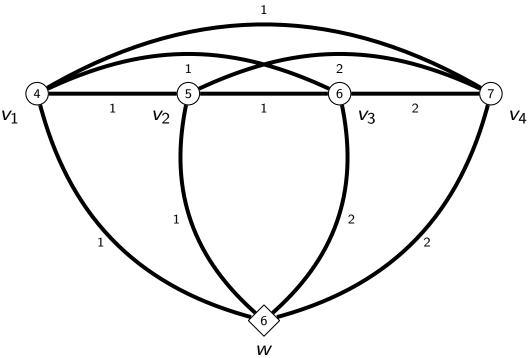

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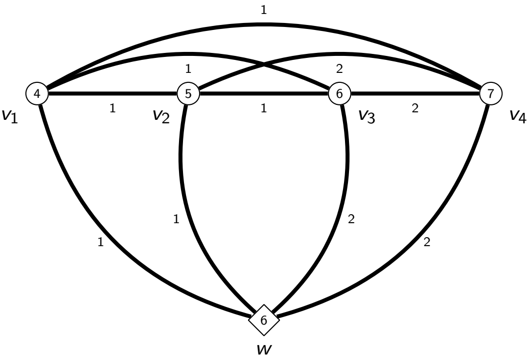

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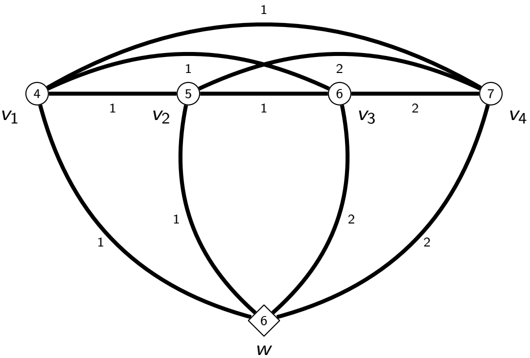

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- we can 2-label so that, also, $\sigma(w) = 6$ $\sim$ half 1's and 2's

Restricting gadgets (cont'd)

Note: locally, we can 2-label the gadget properly, "pushing" conflicts at w
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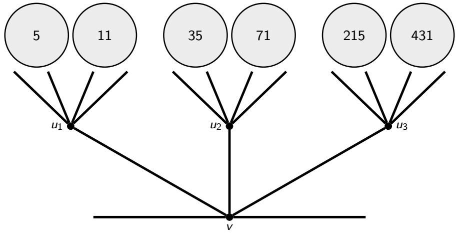
next constructions:

- *attaching a k-clique at v*: add a k-clique, and make it dominated by v
- *attaching, for $k \geq 7$, a k-fan at a degree-2 vertex v*:
  - add $k-2$ vertices $u_1, \dots, u_{k-2}$, adjacent to v
  - attach a k-clique and a $(2k+1)$-clique at u_1; set $n_1 = 3k+2 = d(u_1)$
  - attach a $(2n_1+1)$-clique and a $(2(2n_1+1)+1)$-clique at u_2; set $n_2 = d(u_2)$
  - go on like this for all u_i's one after the other



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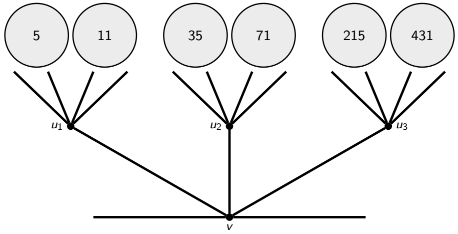

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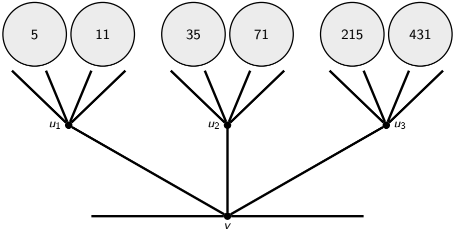

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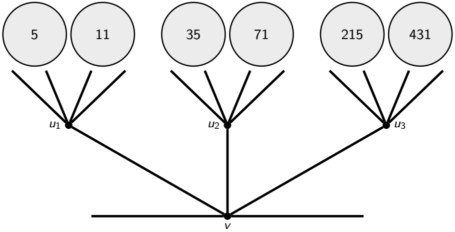

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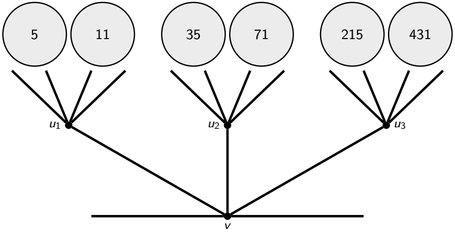

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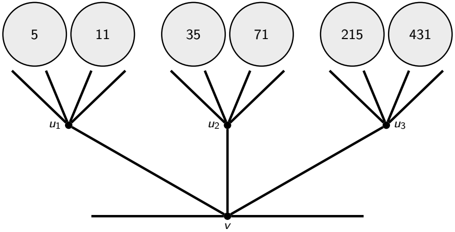

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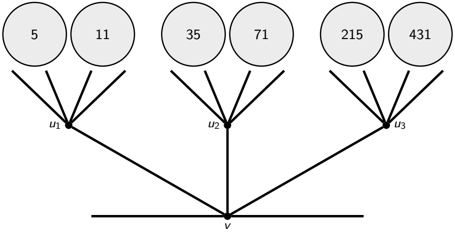

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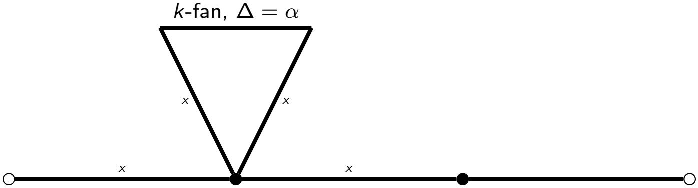

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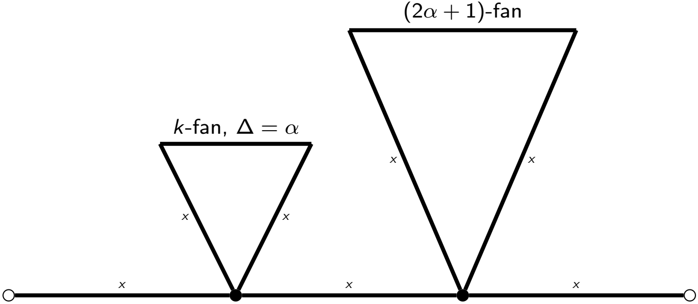

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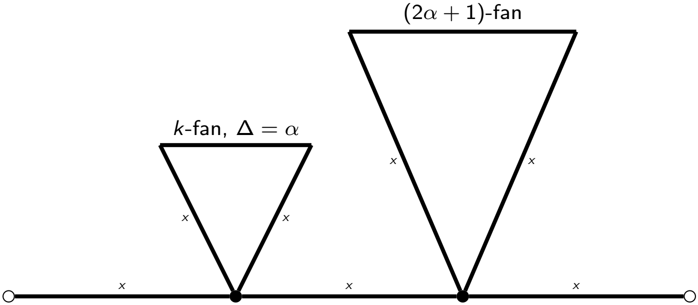

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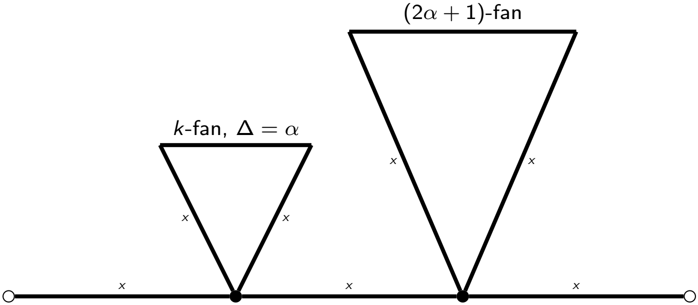

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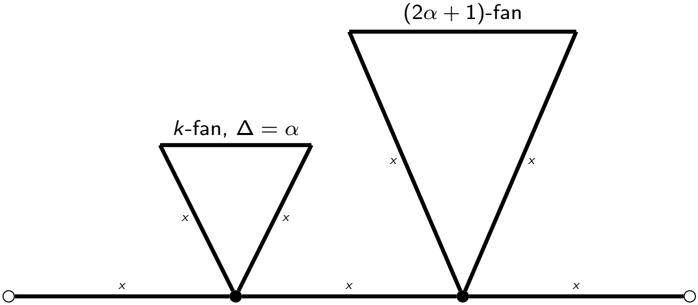

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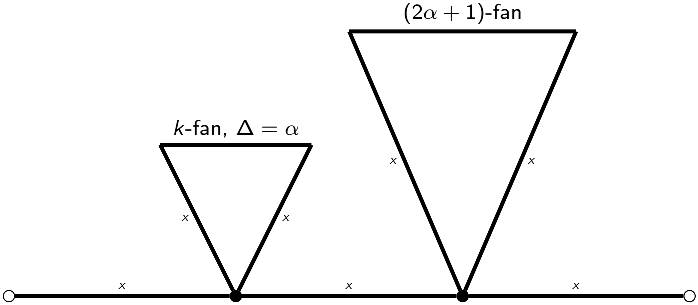

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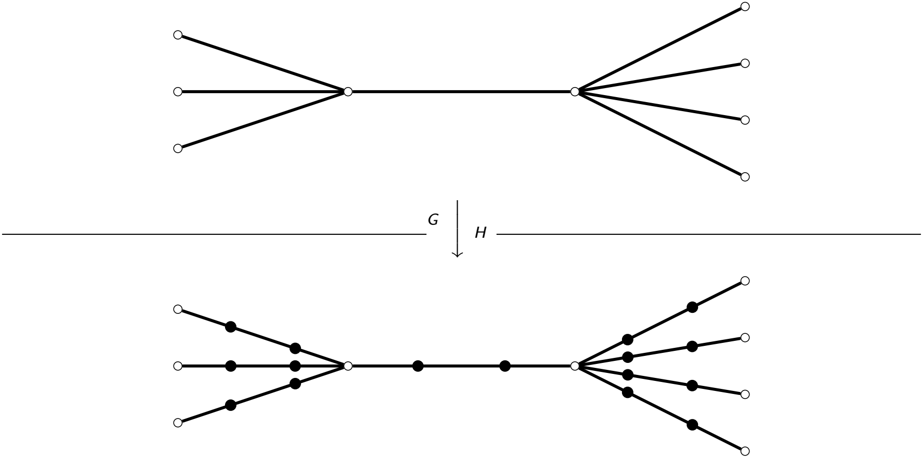

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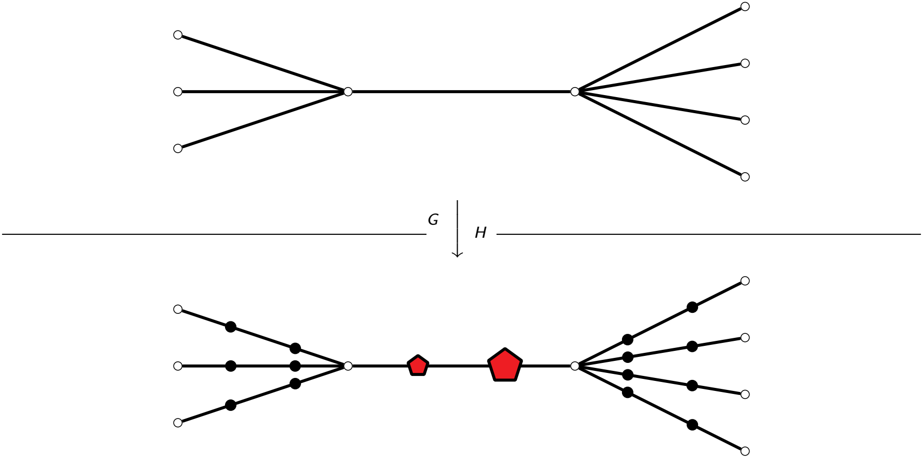

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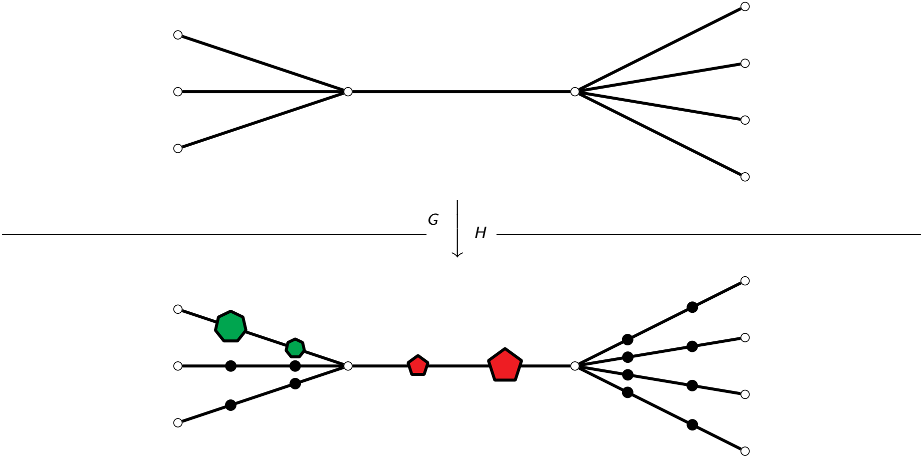

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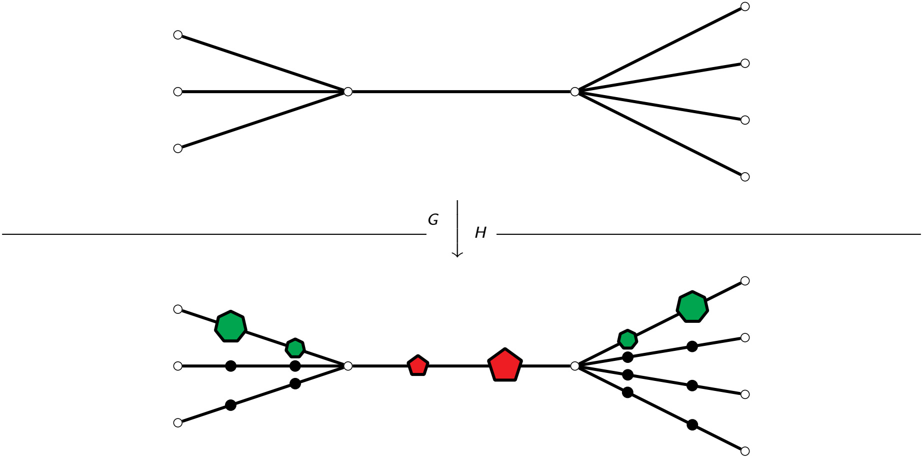

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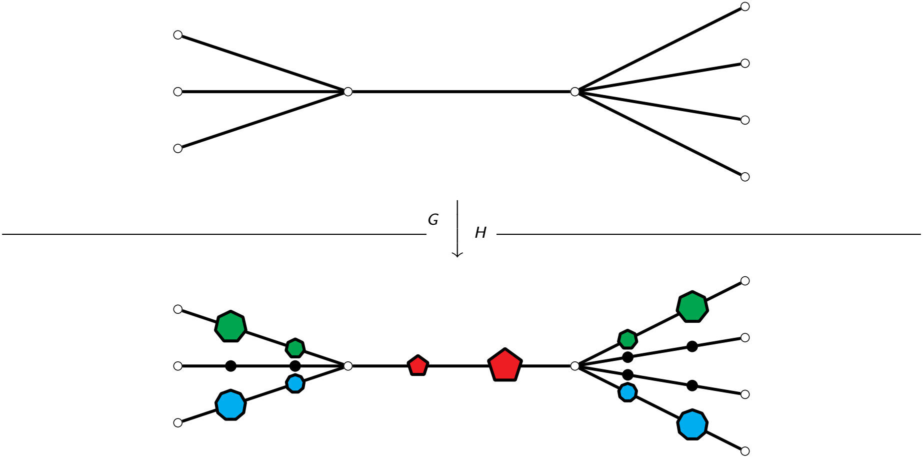

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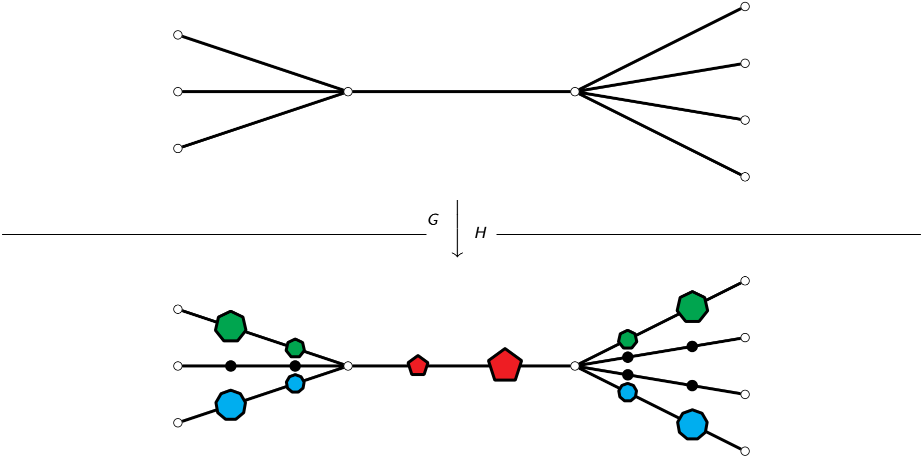

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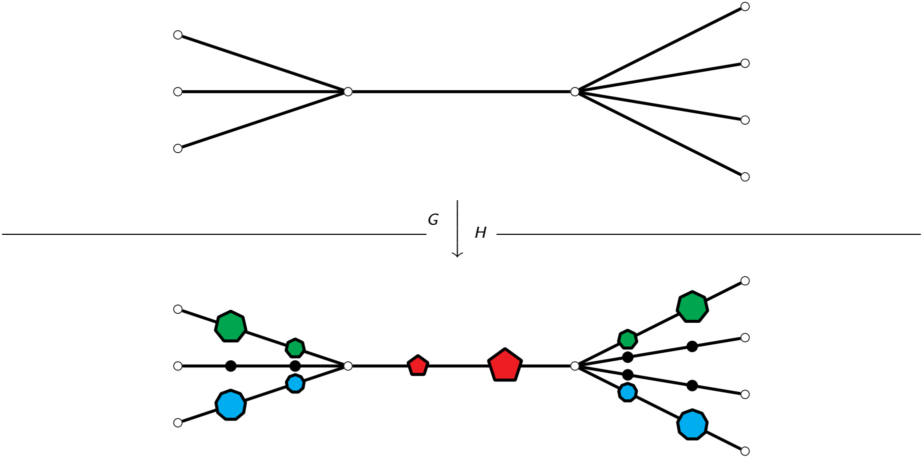

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to limit # of needed types of fans: make sure $\Delta(G)$ is bounded; fortunately:

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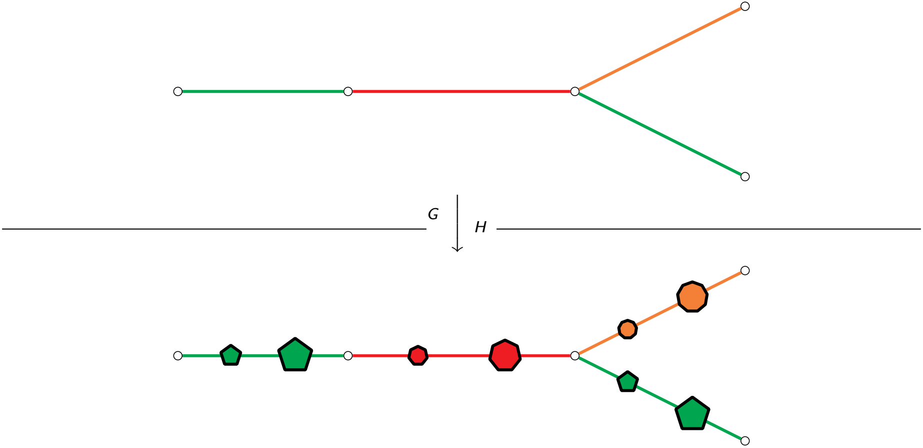
thus, for free, can suppose G comes with a proper 4-edge-colouring ϕ

Coloured fans



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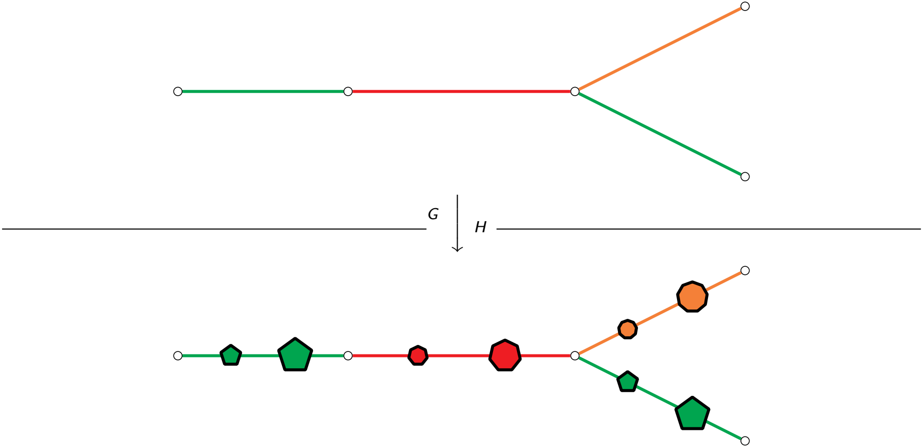

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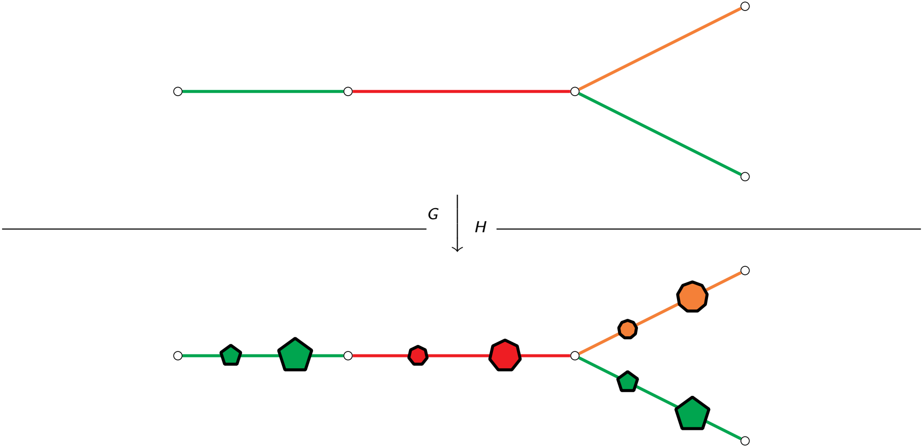

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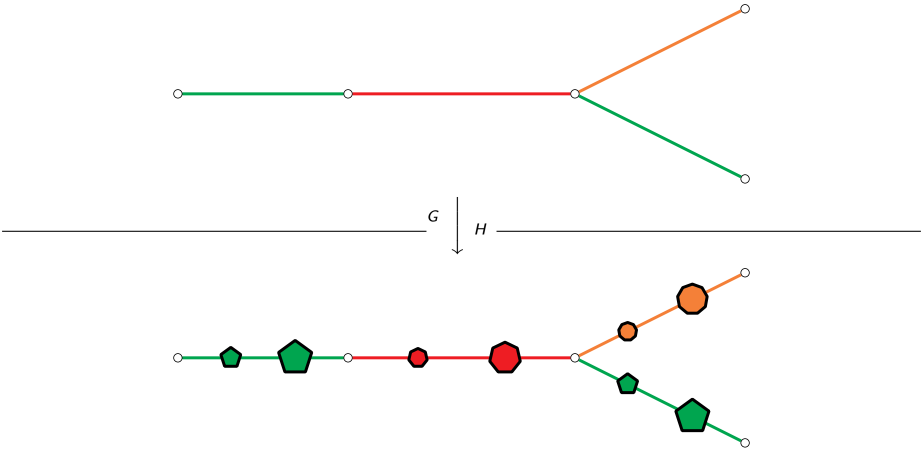

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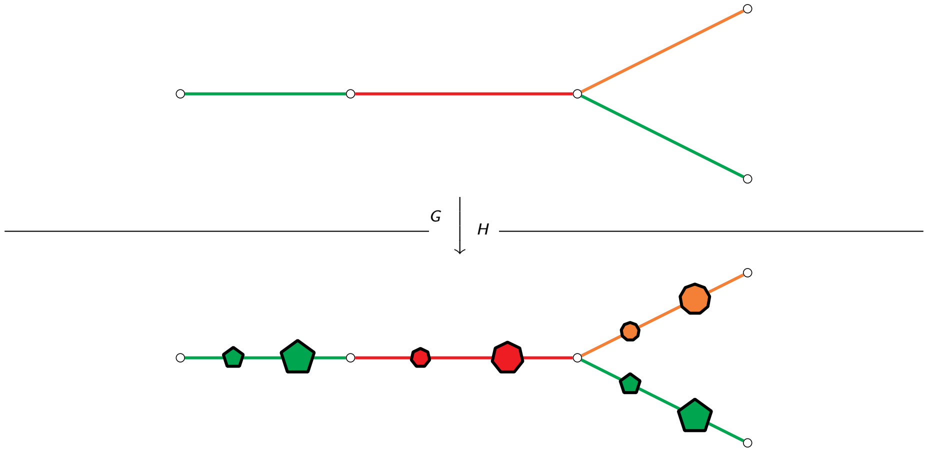
- white vertices have sum at most 6 $\Rightarrow$ no conflicts with fans if (≥ 7)-fans

Coloured fans



now OK ☺ :

- $4(d-1)$ types of fans ⇒ constant number
- ϕ ⇒ fans of the same type are at distance more than d
- white vertices have sum at most 6 ⇒ no conflicts with fans if (≥ 7)-fans

altogether, construction in poly-time + labelling equivalence ☺

Multiset irregularity strength

(Very brief) introduction

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- in particular, **different degrees $\Rightarrow$ different colour codes!**

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Theorem [B., 2022]

For any given graph G, determining whether $s_m(G) \leq 2$ is NP-complete.

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reduction from Monotone Cubic 1-in-3 SAT:

- instance: 3CNF formula F over clauses $C_1, \ldots, C_m$ and variables $x_1, \ldots, x_n$
- all clauses contain exactly three distinct (positive) variables — monotonicity
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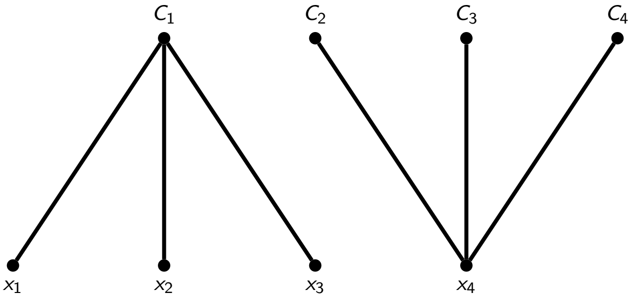
from F, build, in poly-time, graph G, so that F is 1-in-3 satisfiable $\leftrightarrow s_m(G) \leq 2$

Main ideas

1=blue, 2=red

Main ideas: model the structure of F as a graph, and add forcing mechanisms so that reflecting labelling properties (*i.e.*, clause vertices: exactly one blue incident formula edge; variable vertices: three incident formula edges the same colour)



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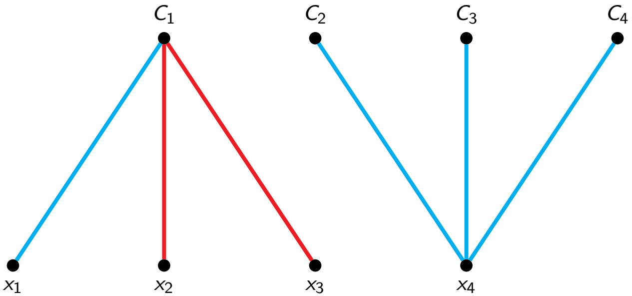

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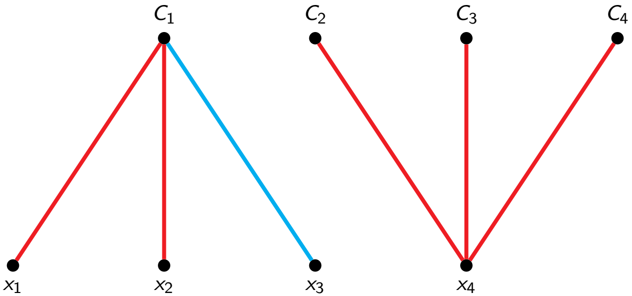

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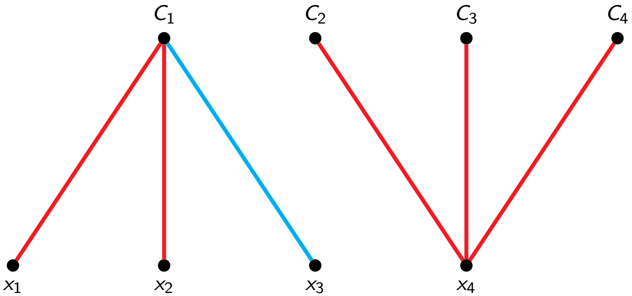
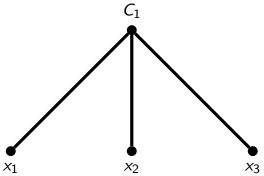


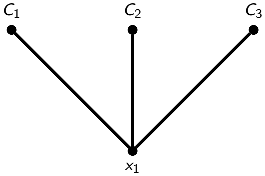
thus, forbid:

- for clause vertices: RRR+RBB+BBB
- for variable vertices: RBB+RRB

Forcing mechanisms

Goal: "generate" vertices with same degree, that must have the forbidden colour codes





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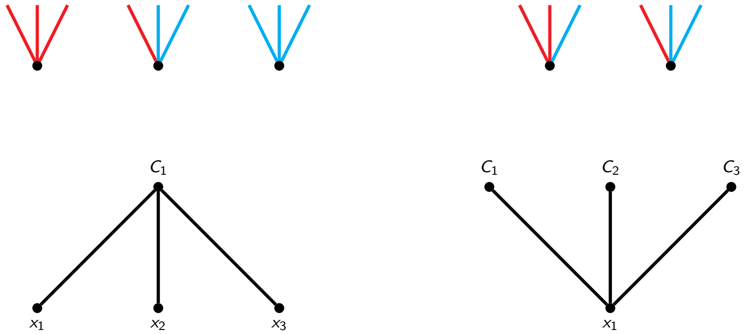

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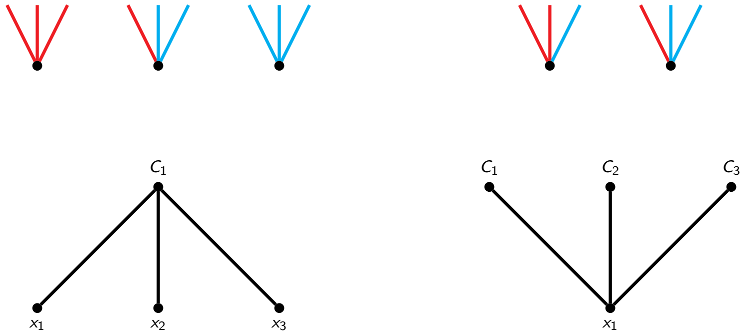

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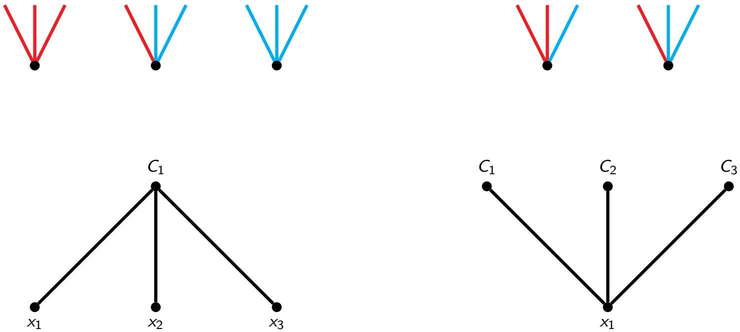

⇒ thus need to "generate" edges of a certain colour

Note: any two variable vertices and/or clause vertices should have distinct degrees...
(for now, let us just pretend ☺ ☺ ☺)

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Note: properties of cliques still apply here
⇒ have, somewhere, a vertex to which forcing cliques are attached:

Implications:

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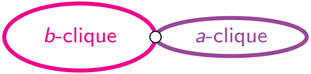

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- *a* vertices of degree *a* + all colour codes but R^a or B^a $\Rightarrow$ only R^a or B^a remains
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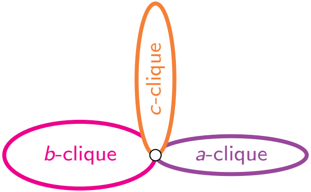

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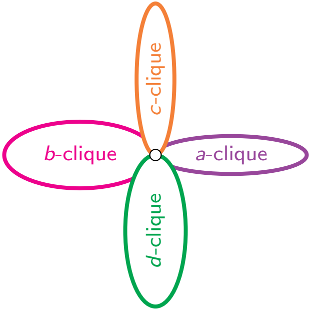

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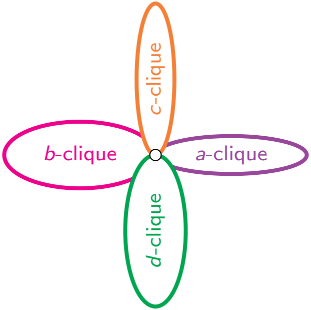

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$\Rightarrow$ for any degree x, can make sure a degree-x vertex is monochromatic

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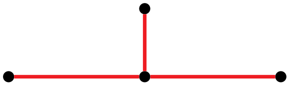

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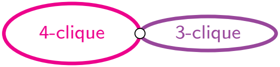

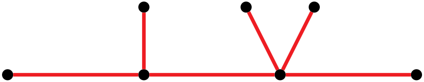

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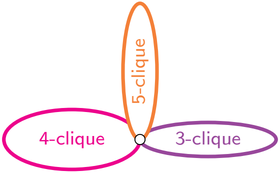

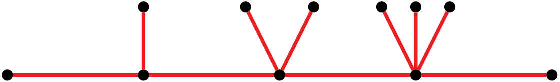

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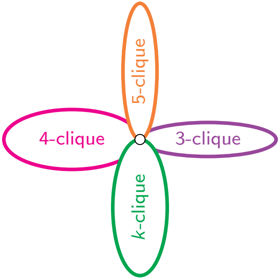

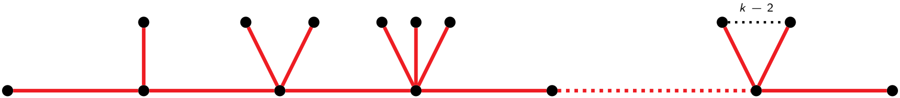

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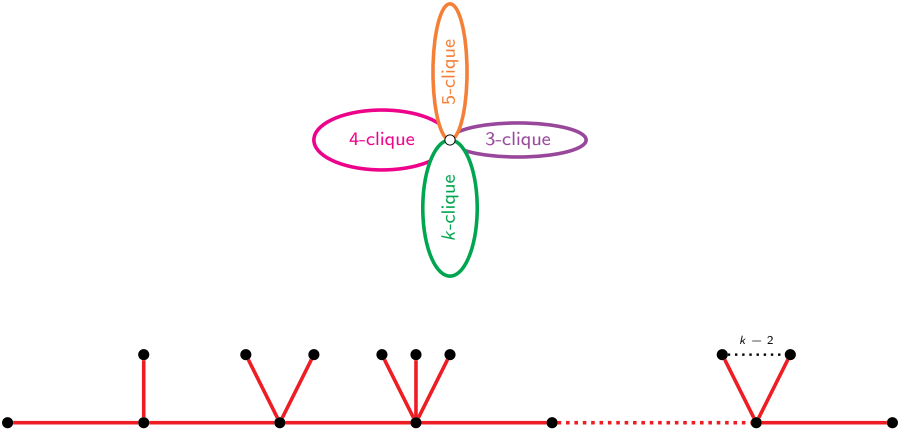
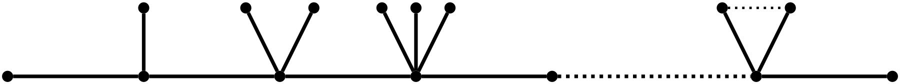
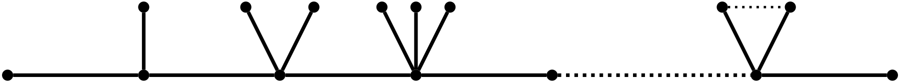


pendant edges: forcing edges + extra edges (discussed later)

Pairs of forcing trails

force two colours: need two forcing trails (with distinct degrees)





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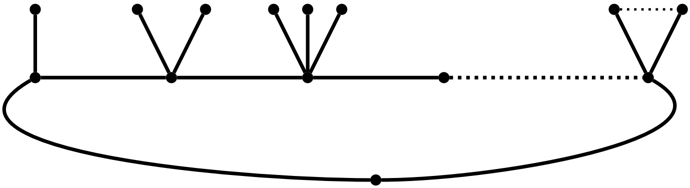

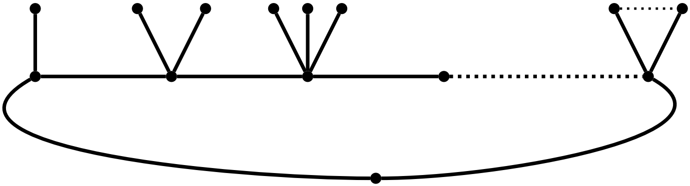

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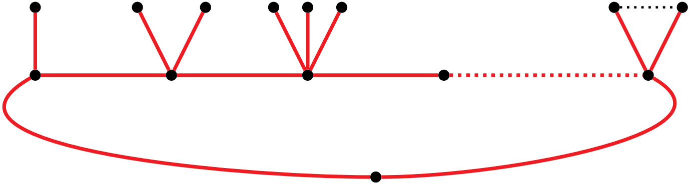

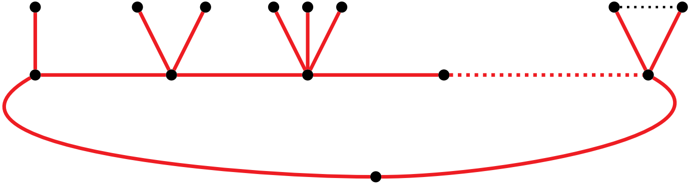

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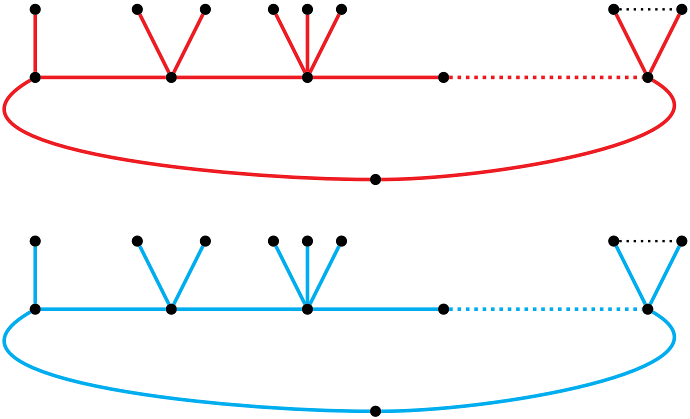

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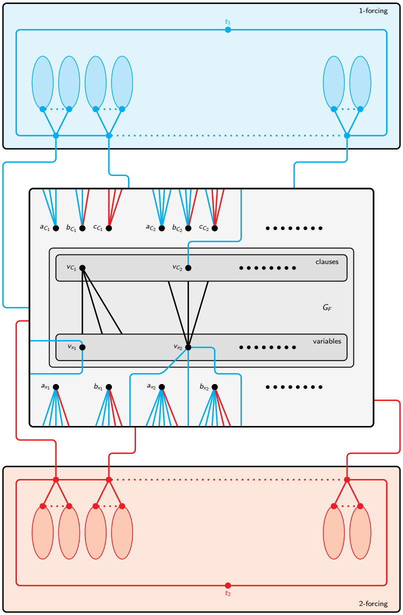
- unused pendant edges: attach cliques (with new degrees) to fill

number of needed degrees: polynomial function of $n, m \Rightarrow$ poly-time construction


1-forcing
t_1
a_{C_1}
b_{C_1}
c_{C_1}
a_{C_2}
b_{C_2}
c_{C_2}
v_{C_1}
v_{C_2}
clauses
G_F
variables
v_{x_1}
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b_{x_1}
a_{x_2}
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t_2
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