A proof of the Multiplicative 1-2-3 Conjecture

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Introduction

The 1-2-3 Conjecture, in few words

"Given a graph, can we assign 1, 2, 3 to its edges, so that
no two adjacent vertices are incident to the same sum of labels?"

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Edge weights and vertex colours

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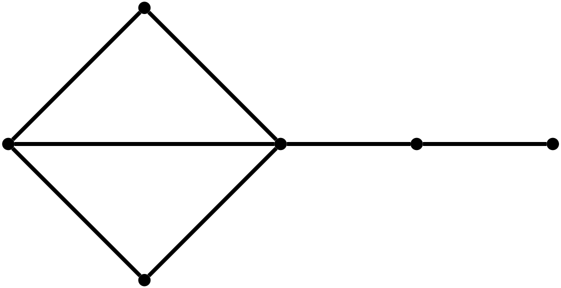
Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?

We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

Sample example



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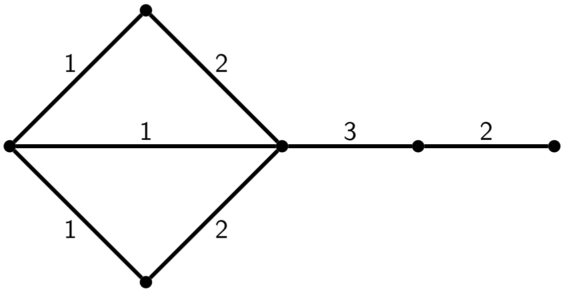

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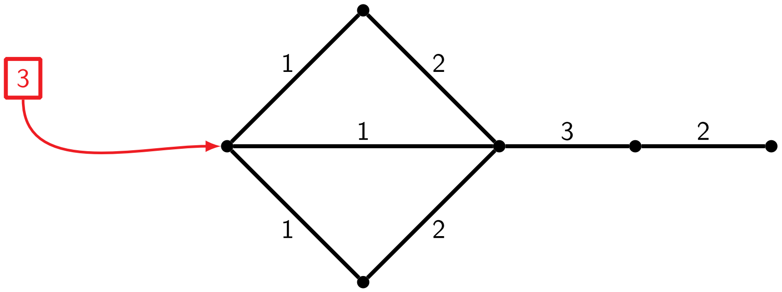

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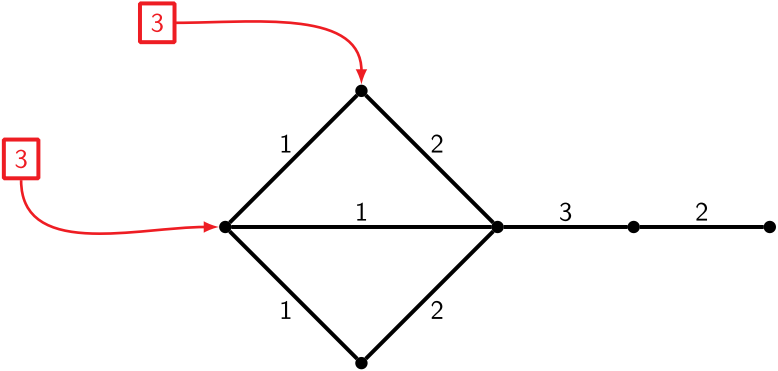

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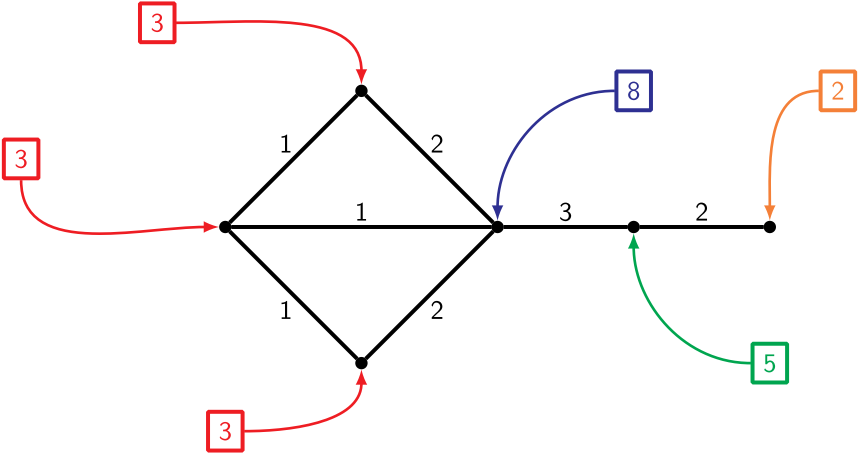

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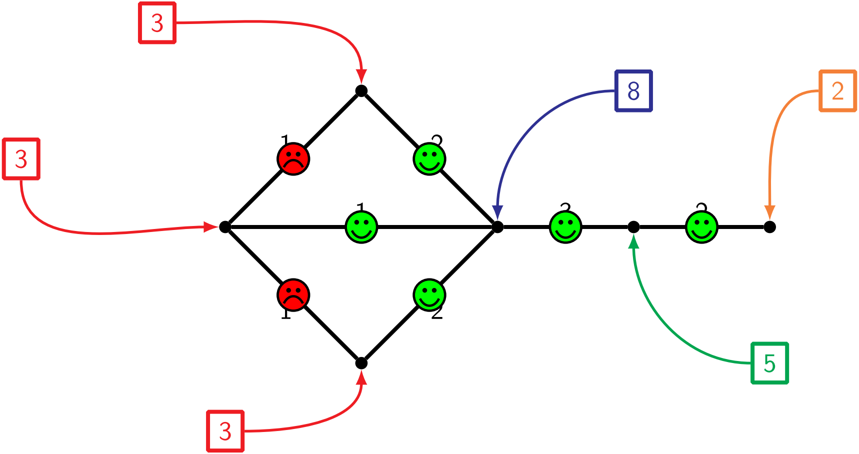

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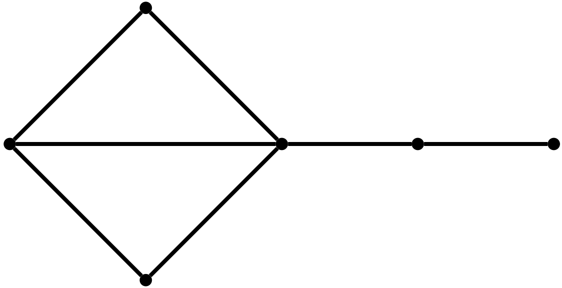

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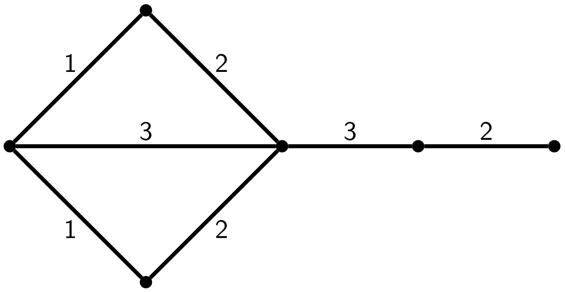

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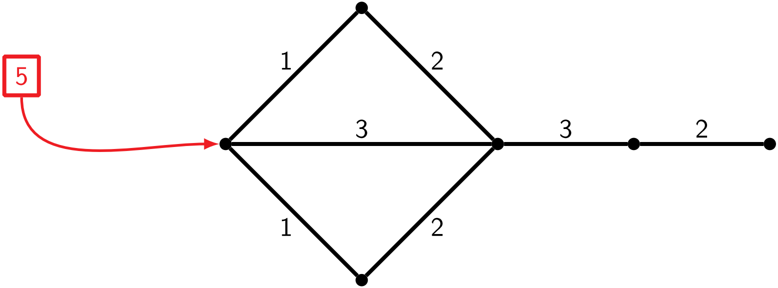

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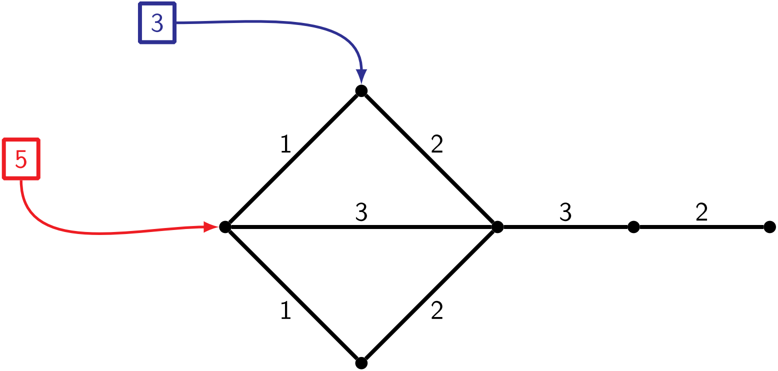

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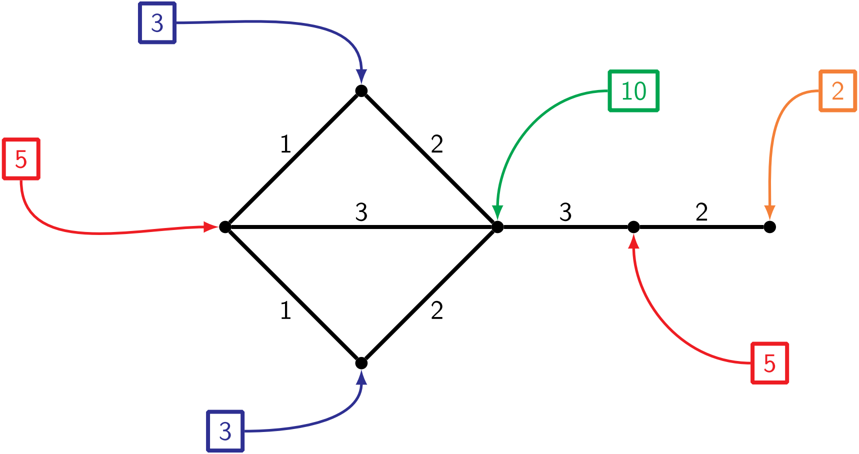

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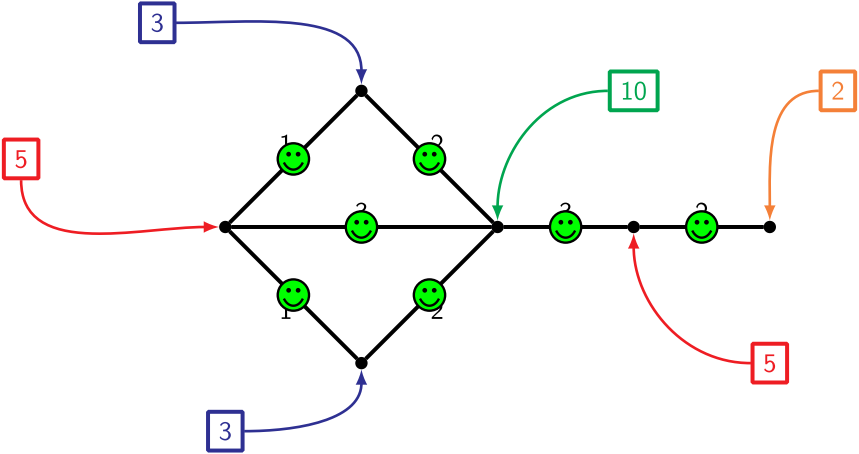

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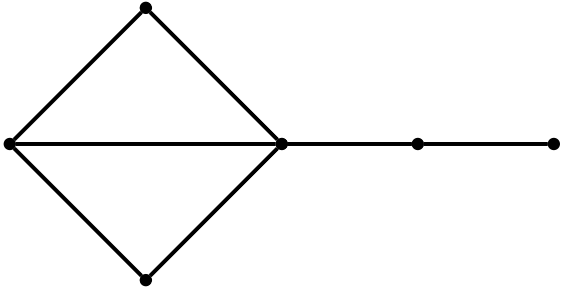

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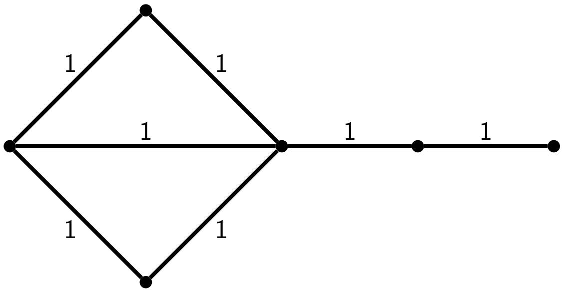

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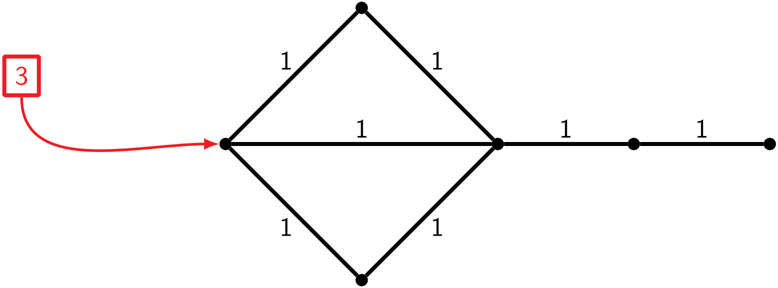

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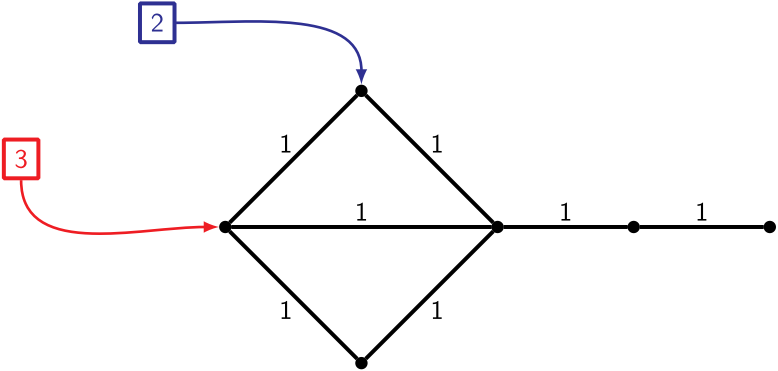

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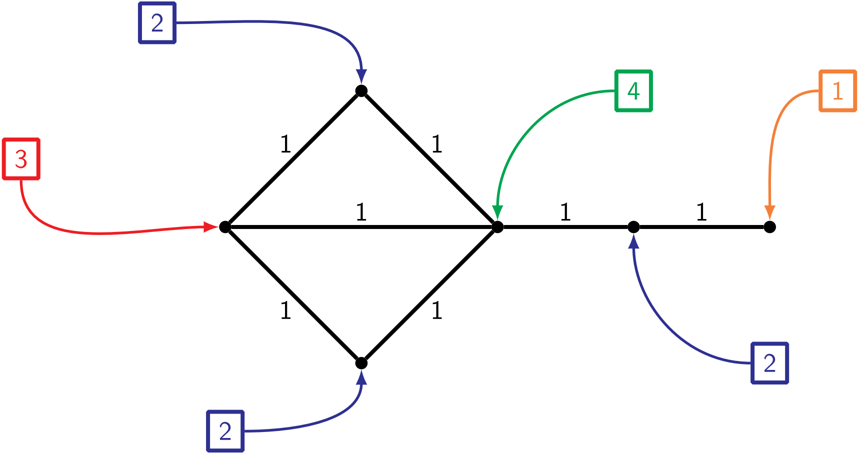

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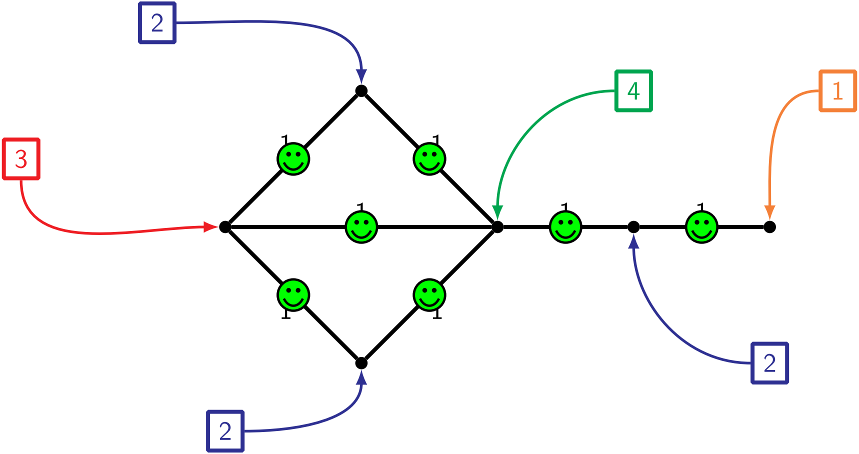

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1-2-3 Conjecture (Karoński, Łuczak, Thomason, 2004)

This is always possible with $k \leq 3$.

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- **Verification of the conjecture:**
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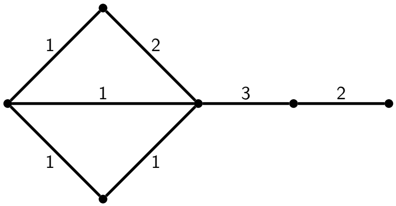
Also, many **side aspects, variants, etc.**

Speaking of variants...

- **Multiset variant**



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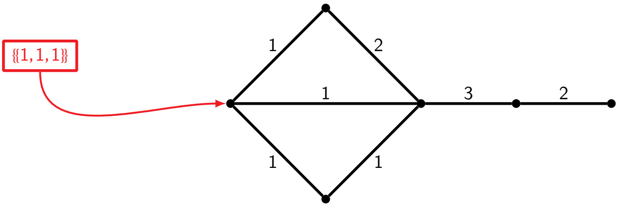

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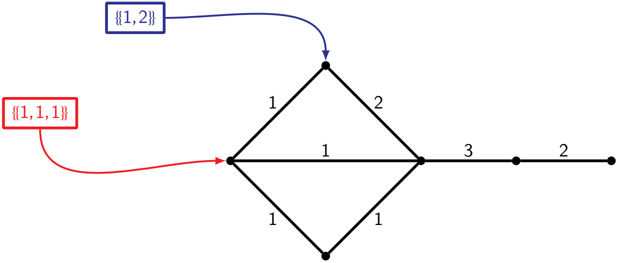

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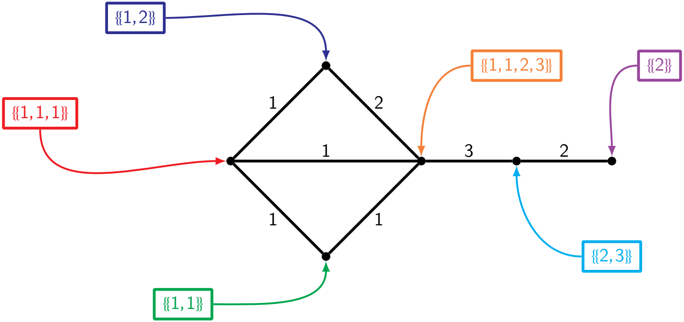

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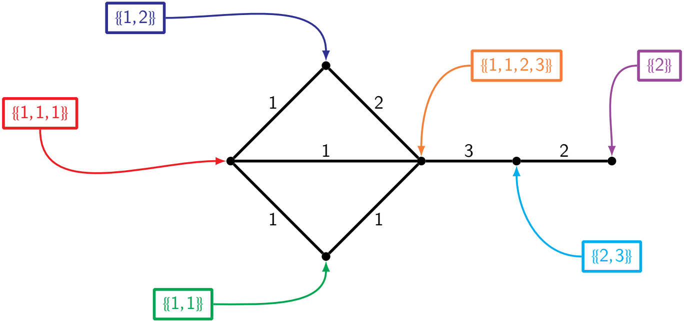

1-2-3 Conjecture, multiset version (Addario-Berry *et al.*, 2005)

Labels $1, 2, 3$ suffice for all graphs.

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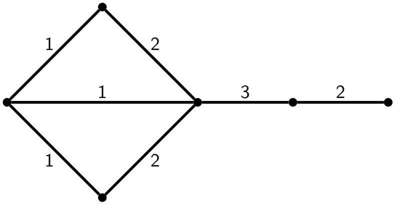

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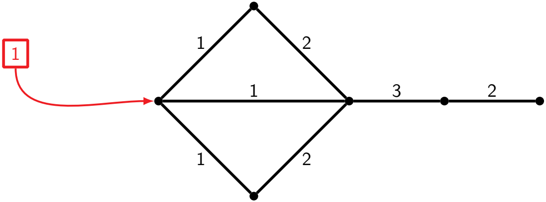

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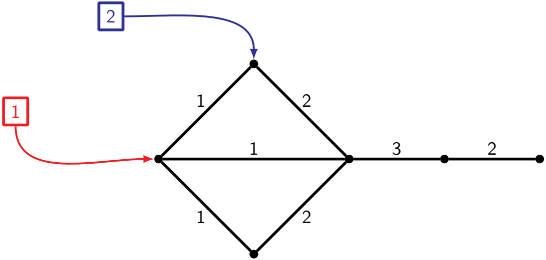

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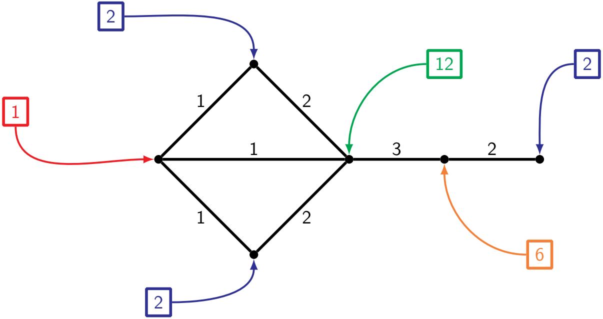

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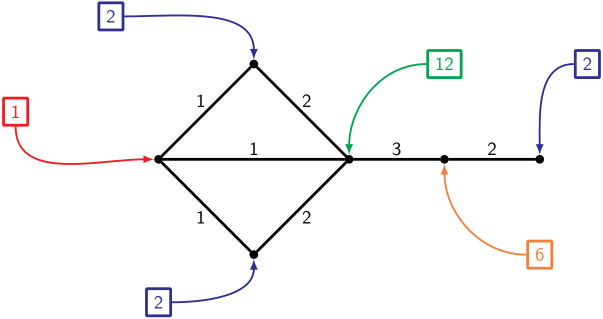

1-2-3 Conjecture, product version (Skowronek-Kaziów, 2012)

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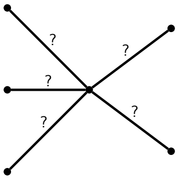
Comparing the three variants

Labels are anything in $\{1, 2, 3\}$



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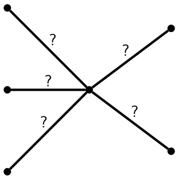

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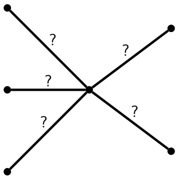

If I tell you:

- sum is 10 ☹ ☹ ☹
- product is 18 😐...

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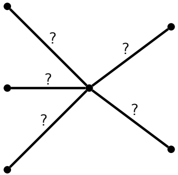

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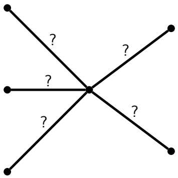

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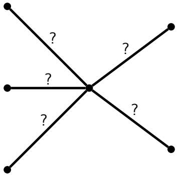

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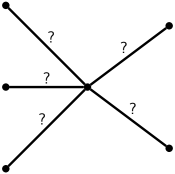

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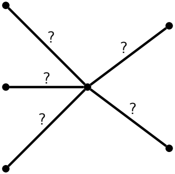

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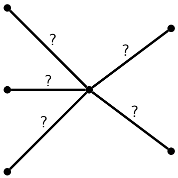

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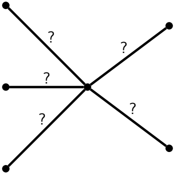

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$\Rightarrow$ **product version $\sim$ multiset version with a neutral label**

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For today: most of the proof!

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Remark: no conflict between

- i-monochromatic and j-monochromatic for $i \neq j$
- monochromatic and bichromatic

Actually, conflict between $2^p 3^q$ and $2^{p'} 3^{q'}$ iff $p = p'$ and $q = q'$

Some terminology and conventions

For any 3-labelling ℓ:

- / = 1, / = 2, / = 3 (**Note:** / and / can be interchanged)
- 2-degree $d_2(v)$ = number of 2's incident
- 3-degree $d_3(v)$ = number of 3's incident
- G = 1-*monochromatic* (product is 1)
- R = 2-*monochromatic* (product is 2^p for $p > 0$)
- B = 3-*monochromatic* (product is 3^q for $q > 0$)
- *bichromatic* = product is $2^p 3^q$ for $p, q > 0$

Remark: no conflict between

- i-monochromatic and j-monochromatic for $i \neq j$
- monochromatic and bichromatic

Actually, conflict between $2^p 3^q$ and $2^{p'} 3^{q'}$ iff $p = p'$ and $q = q'$

- S = *special* (product is $2^{2p} 3$ for $p > 0$)

Sketch of the proof

Main labelling steps

Start from all edges labelled 1

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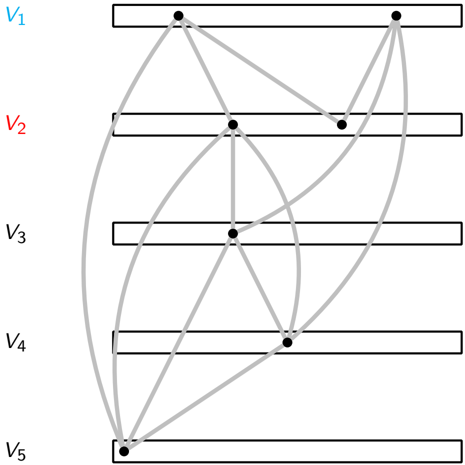
The type of labelling we want by the end of Step 2



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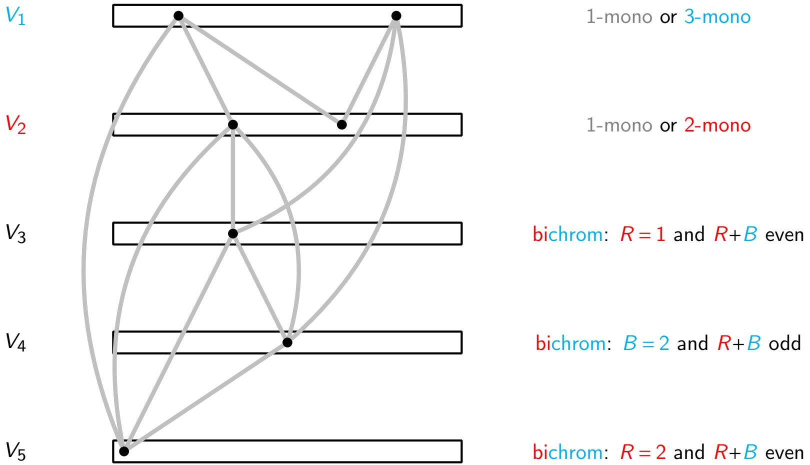

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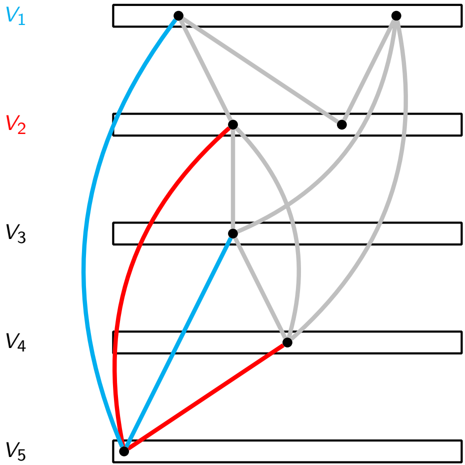

V_1: 1-mono or 3-mono

V_2: 1-mono or 2-mono

V_3: bichrom: $R = 1$ and $R+B$ even

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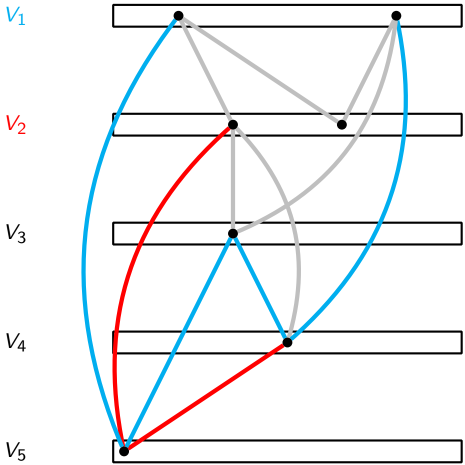

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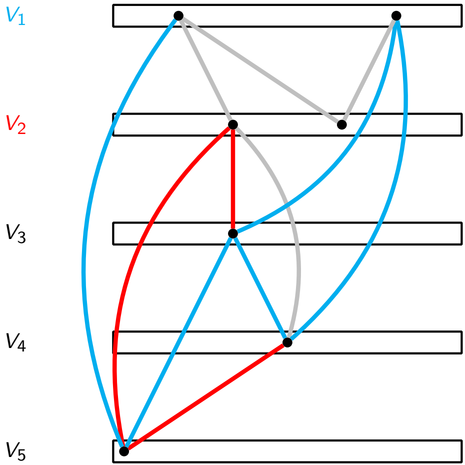

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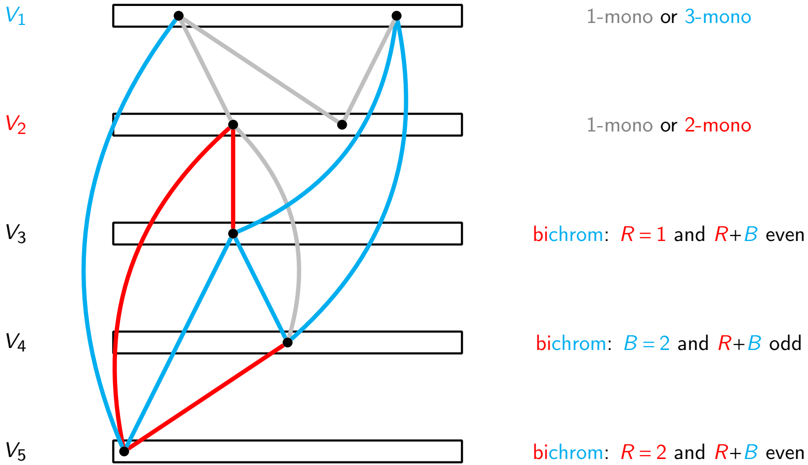
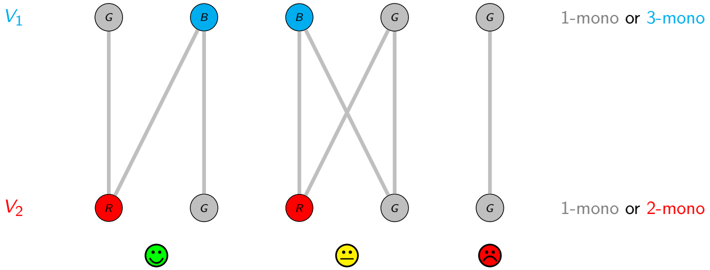


Note:

- no conflict between odd layers; same for even layers
- same between odd layers and even layers (except for 1-mono across (V_1, V_2))
- no special vertex ($B = 1$ and $R + B$ odd)

Getting rid of remaining conflicts in Step 3



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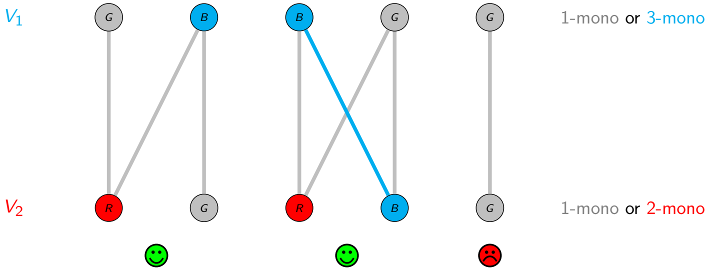

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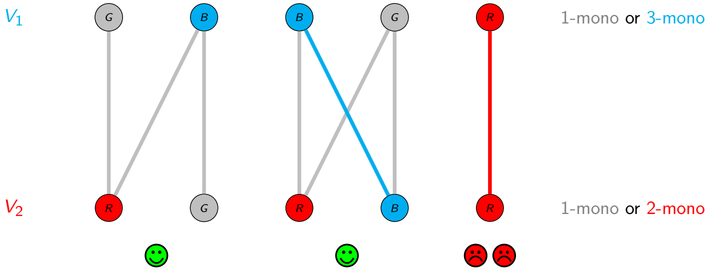

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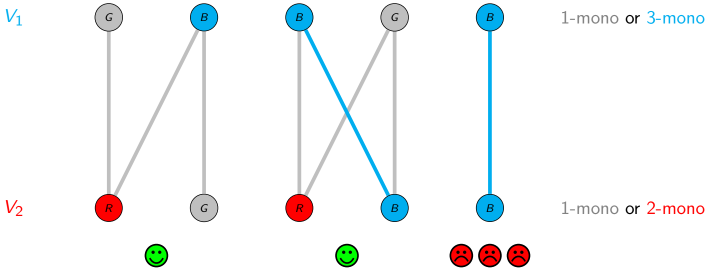

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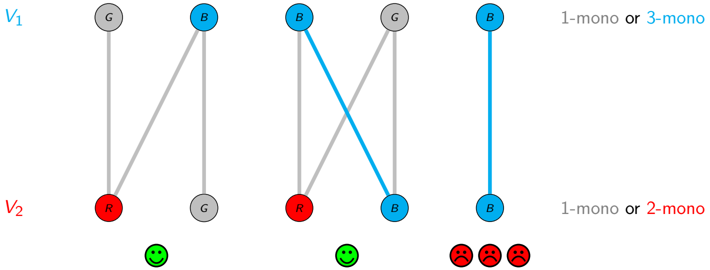

Do not forget about $V_3, \ldots, V_t$!!

⇒ Keep vertices 1-mono, 2-mono, 3-mono, special

Main labelling steps – REVISITED

Start from all edges labelled 1

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   - **no isolated 1-mono edge in (V_1, V_2)**
3. Get rid of conflicts in (V_1, V_2), playing with 1-mono, 2-mono, 3-mono, special

– Step 1 –

Getting a “good” partition $V_1 \cup \cdots \cup V_t$ of $V(G)$

Getting the upward edges property

- Pick V_1 independent **as big as possible**

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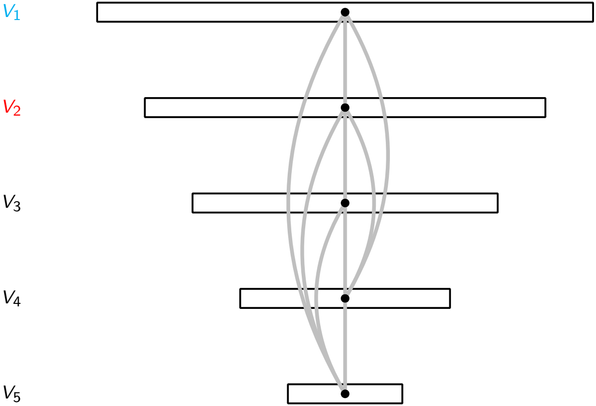

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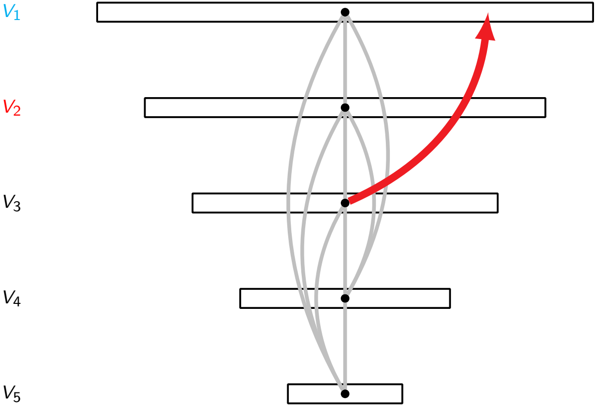

An additional swapping property

Lemma

We can choose $V_1 \cup \cdots \cup V_t$ so that if $e = (u, v) \in (V_1, V_2)$ is isolated, then u and v can be freely exchanged between V_1 and V_2 without spoiling any of the desired properties (independence, upward edges, etc.).

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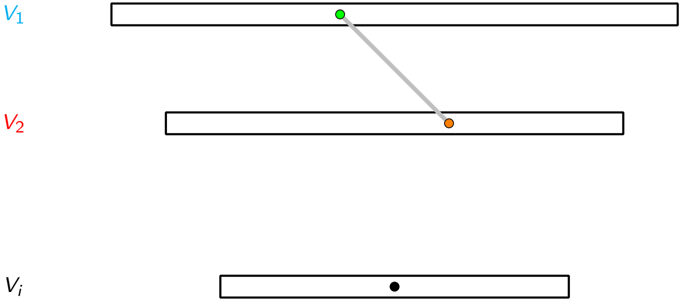
Just choose V_1 and V_2 so that $V_1 \cup V_2$ as large as possible



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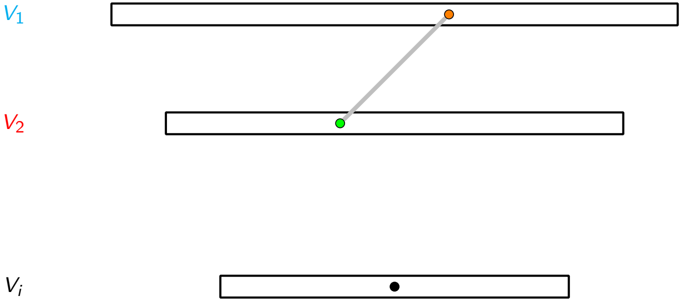

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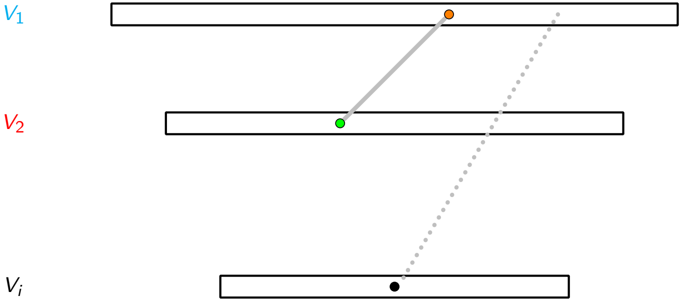

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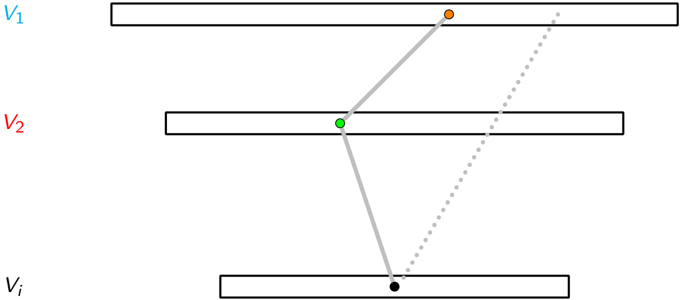

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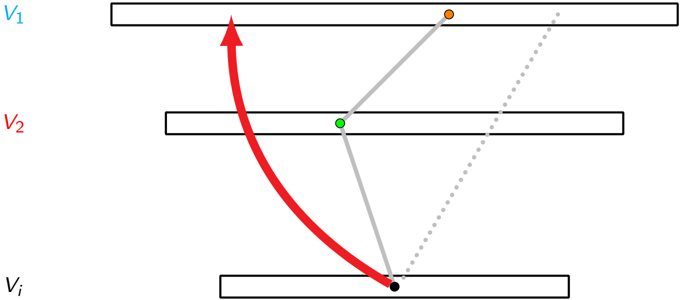
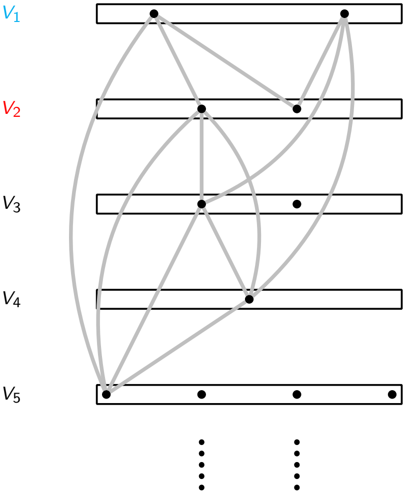


– Step 2 –

Relabelling the upward edges of $V_3, \ldots, V_t$

Recap of what is desired



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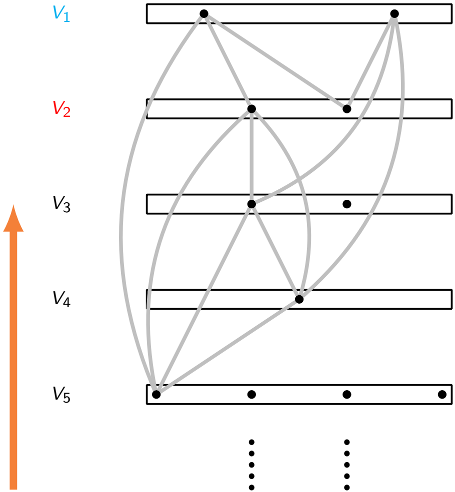

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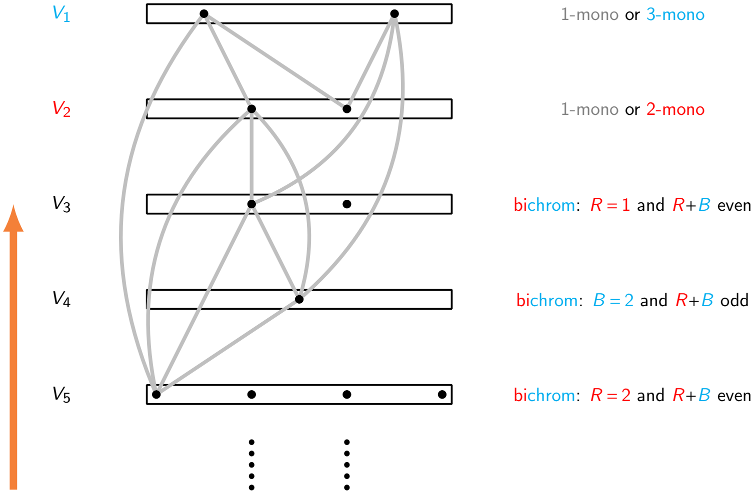
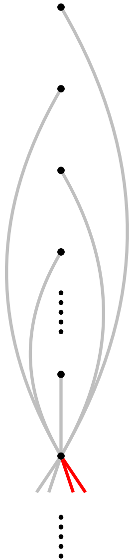


Watch out: even (odd, resp.) layers require a bounded number of 3's (2's, resp.)
⇒ even (odd, resp.) layers produce their 3's (2's, resp.) upwards
⇒ assume even (odd, resp.) layers do not receive 3's (2's, resp.) from below

Case of a vertex in some V_{2n}



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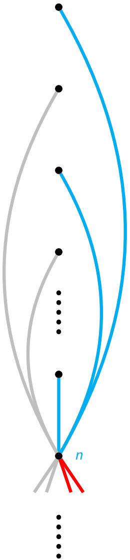

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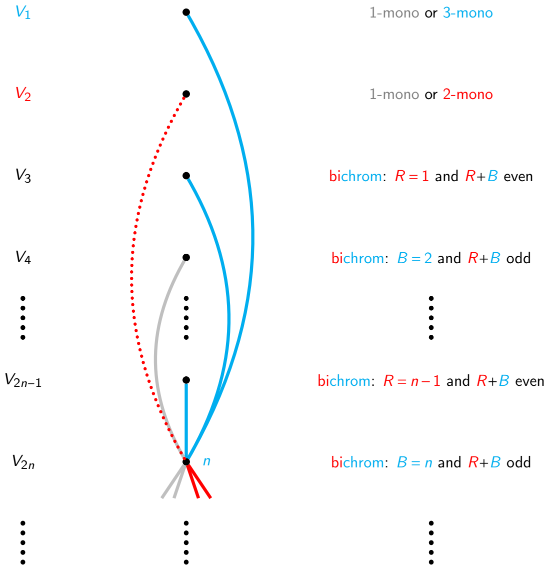
Case of a vertex in some V_{2n} – fixing parity



Works because...

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  $\Rightarrow$ get the required fixed number of labels (3 for even layers, 2 for odd layers)

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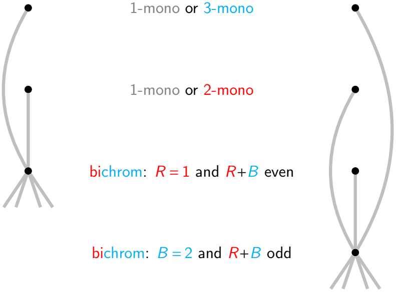

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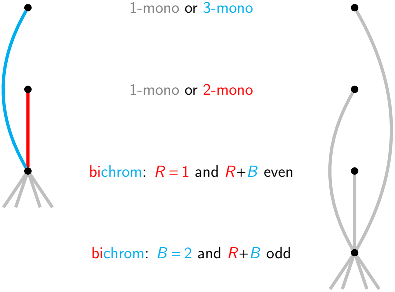

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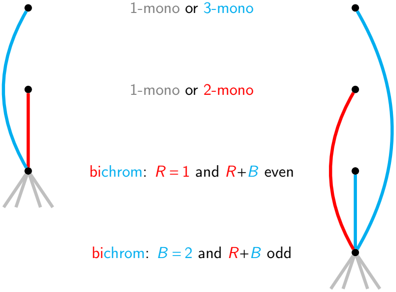
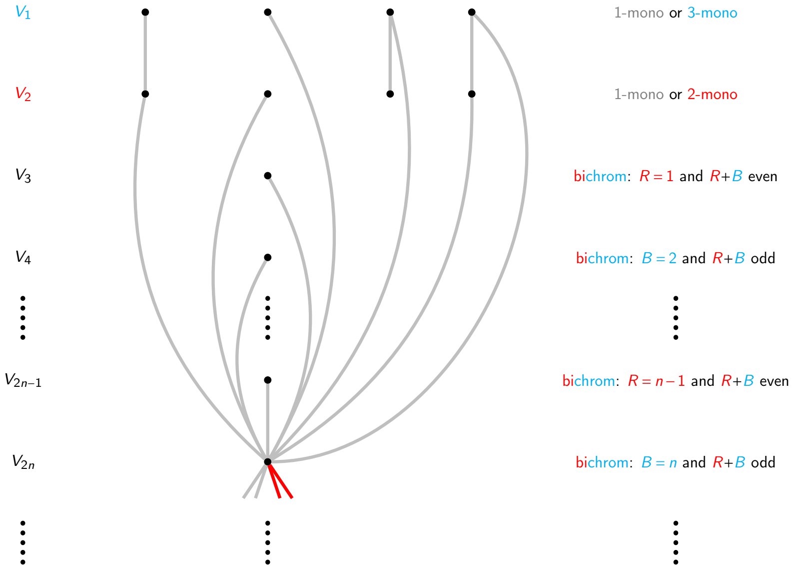
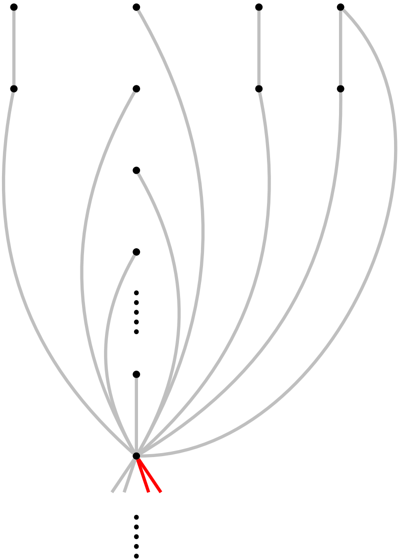


Watch out for adjacent isolated edges in (V_1, V_2)!



Swapping adjacent isolated edges



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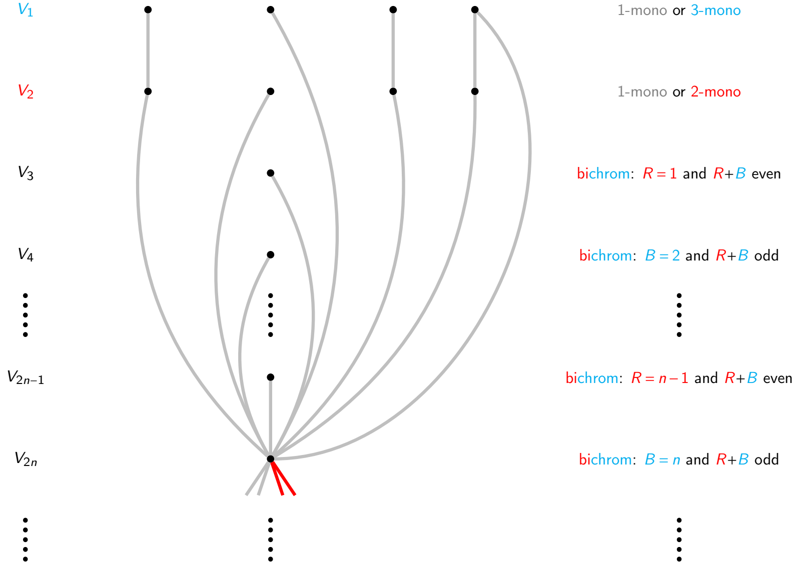
Making adjacent isolated edges happy



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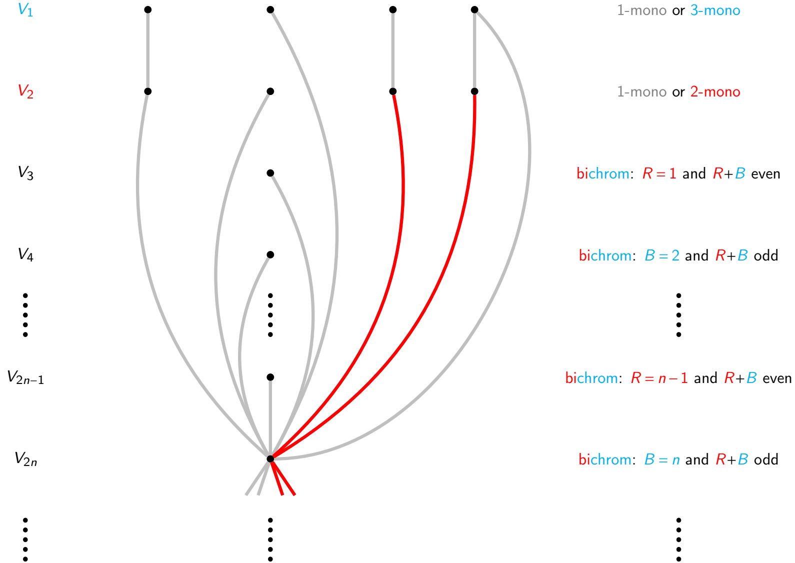

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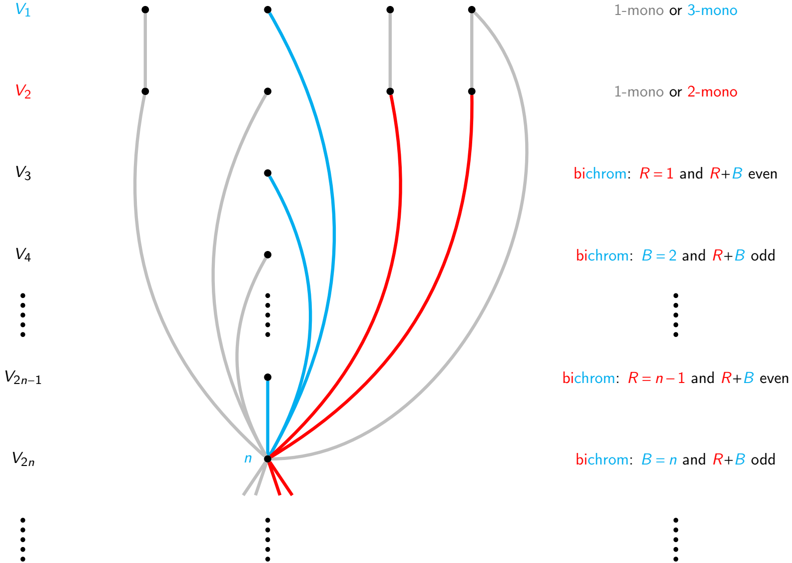
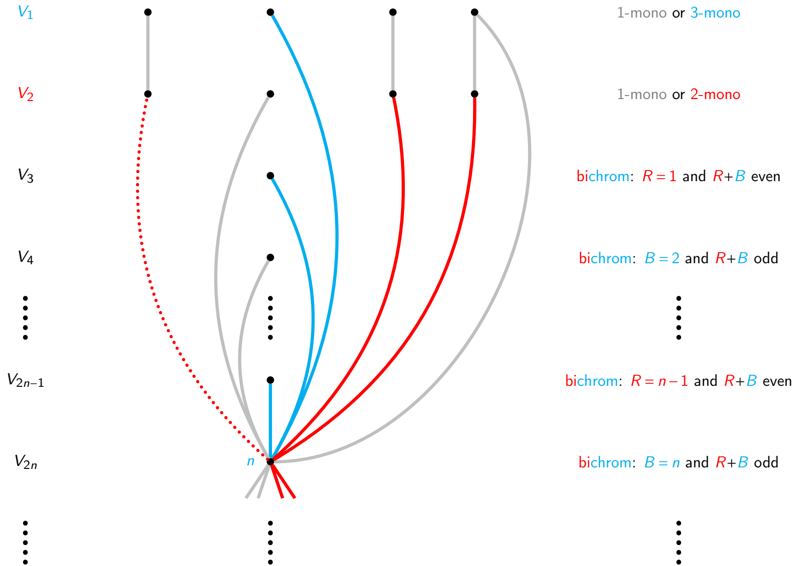


Making adjacent isolated edges happy – fixing parity



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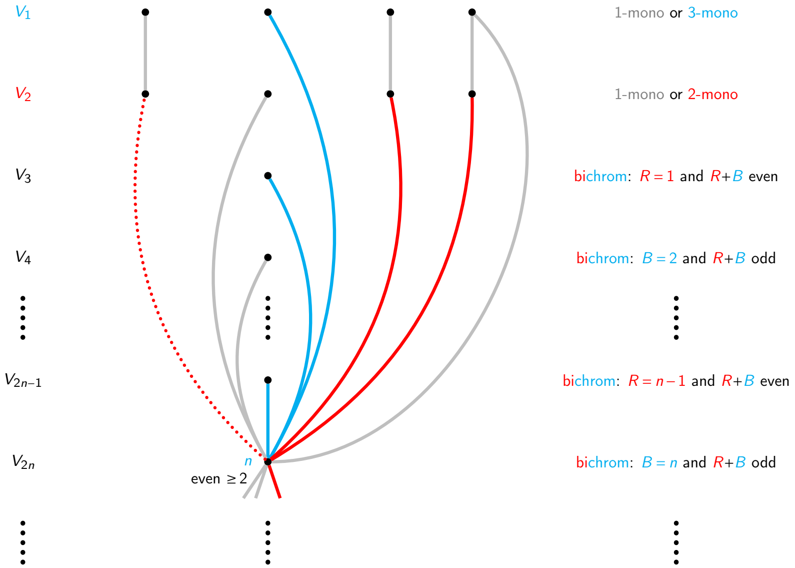

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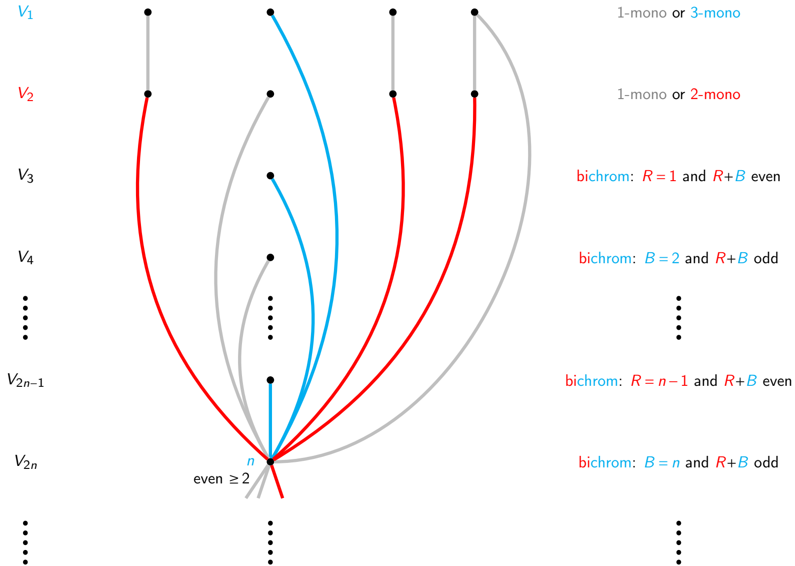

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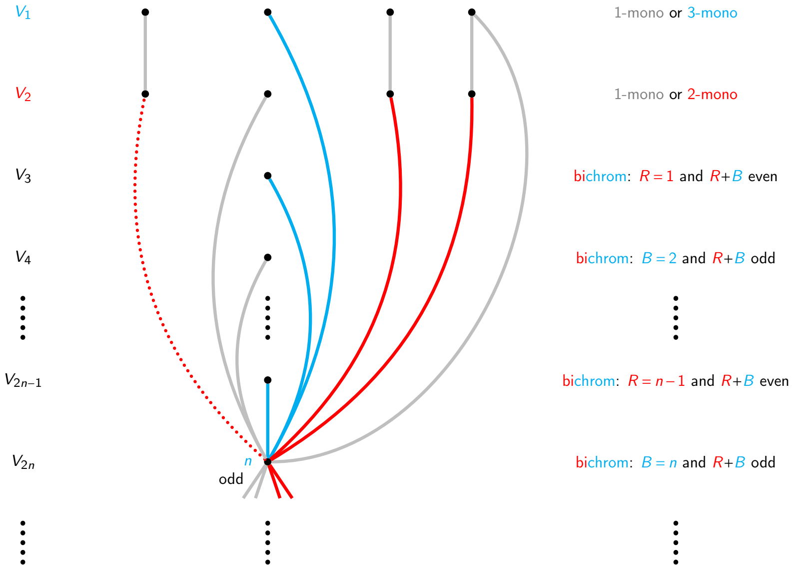

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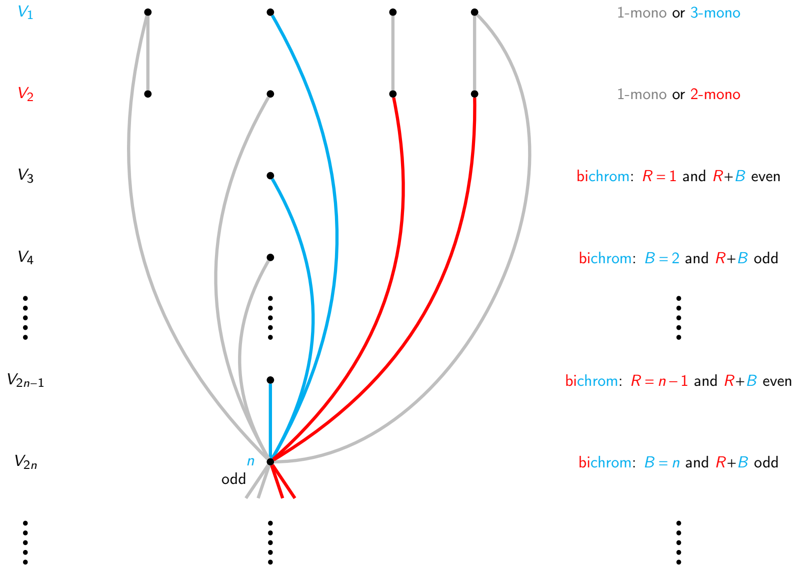

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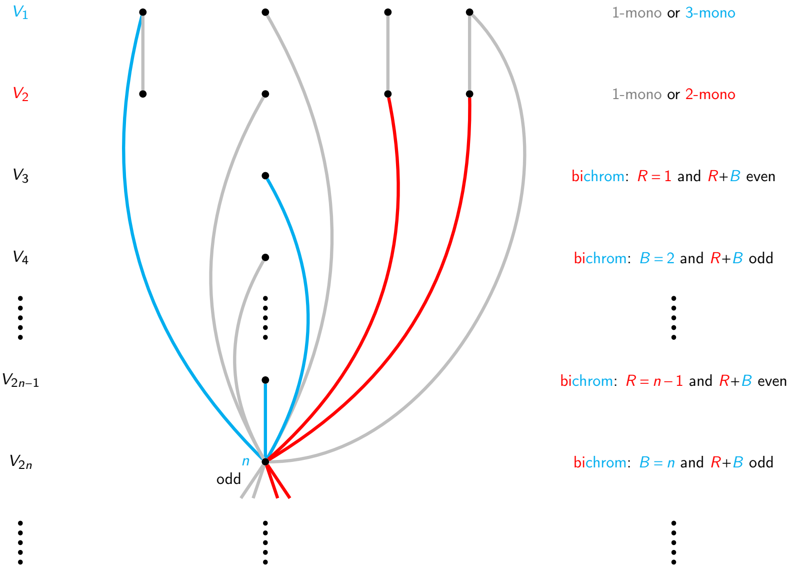
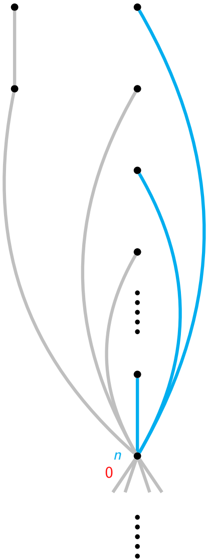
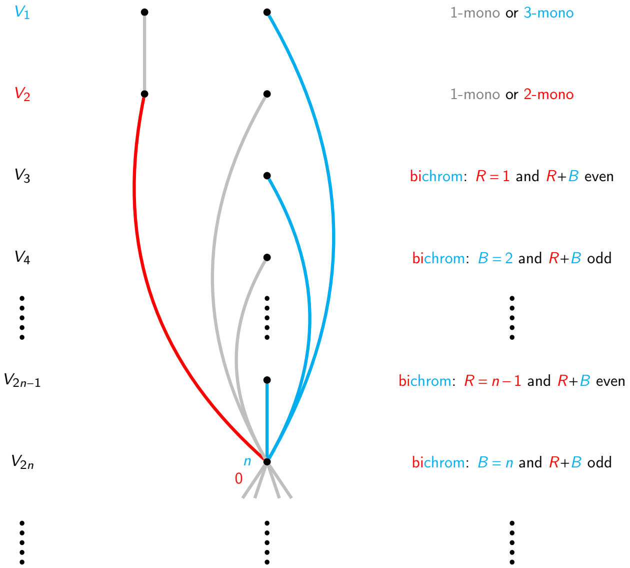


Making adjacent isolated edges happy – fixing parity



V_1	1-mono or 3-mono
V_2	1-mono or 2-mono
V_3	bichrom: $R=1$ and $R+B$ even
V_4	bichrom: $B=2$ and $R+B$ odd
⋮	⋮
V_{2n-1}	bichrom: $R=n-1$ and $R+B$ even
V_{2n}	bichrom: $B=n$ and $R+B$ odd
⋮	⋮

Making adjacent isolated edges happy – fixing parity



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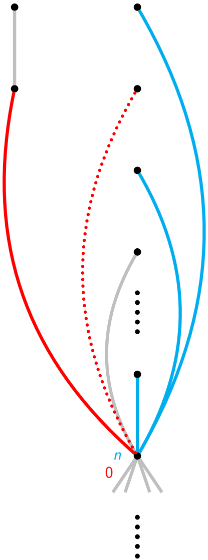

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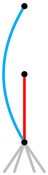
More examples – The (slightly more) intricate case of V_3

Want: bichrom: $R=1$ and $R+B$ even



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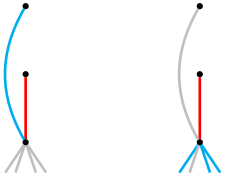

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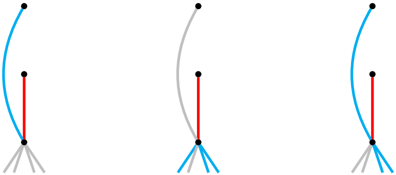

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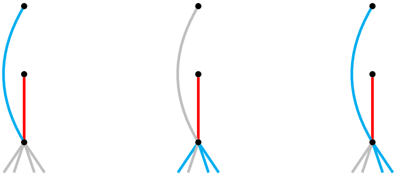

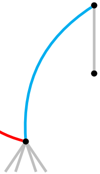

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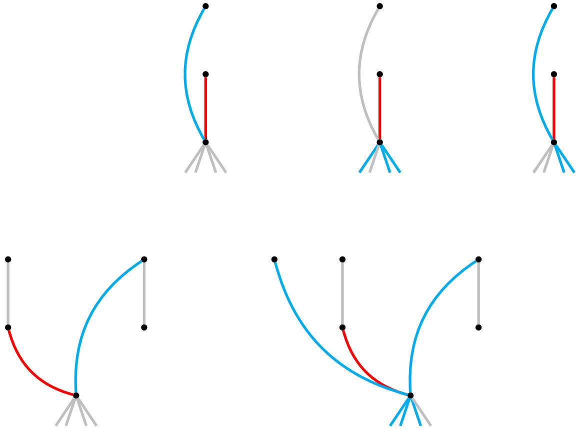

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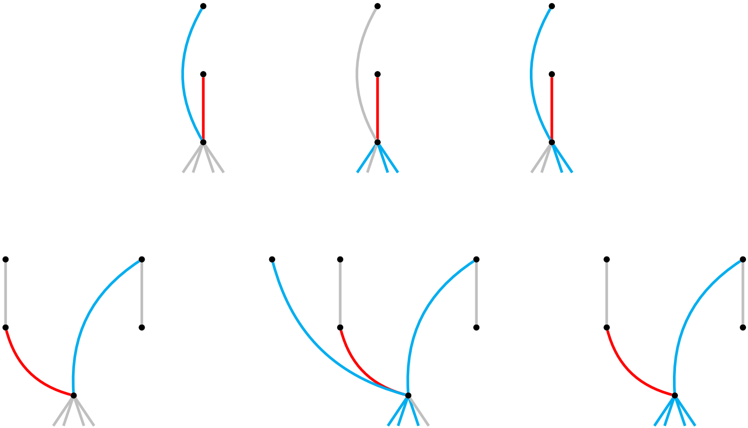
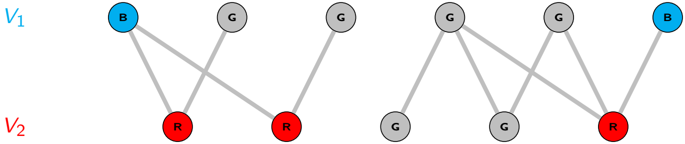


– Step 3 –

Getting rid of conflicts in (V_1, V_2)

Getting rid of conflicts one by one

$\mathscr{H}$: components of $G[V_1 \cup V_2]$ having conflicting (1-mono) vertices



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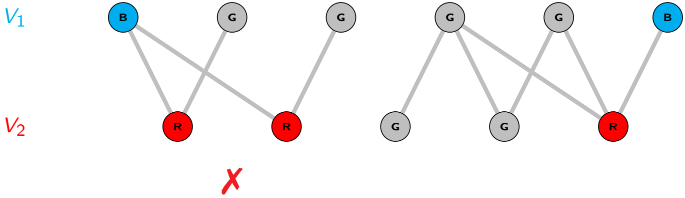

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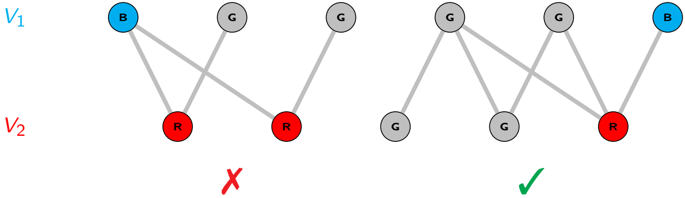

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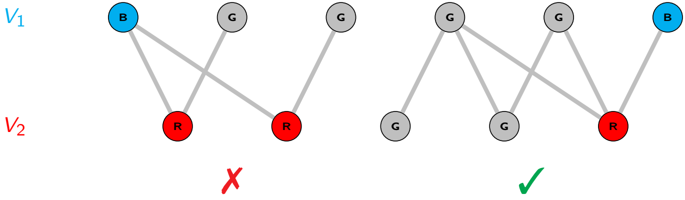

Deal with every $H \in \mathscr{H}$:

- 1-mono, 2-mono, 3-mono, special ⇒ no conflicts with $V_3, \ldots, V_t$
- **Remark:** H's can be treated independently

Getting rid of conflicts one by one

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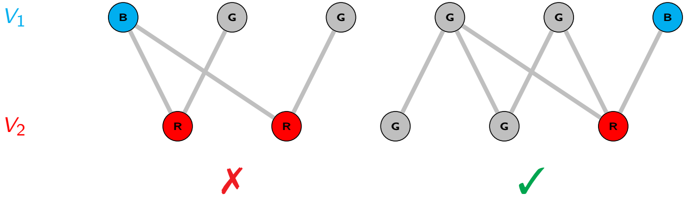

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Several cases:

1. H contains either:
    - a 1-mono $v_1 \in V_1$ with two 1-mono degre-1 neighbours $u_1, u_2 \in V_2$, or
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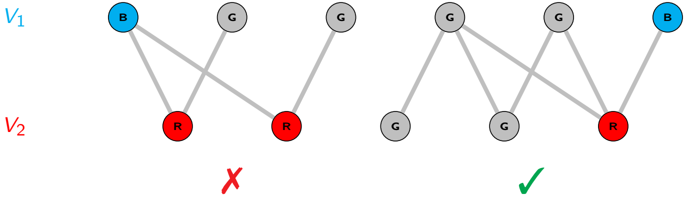
Getting rid of conflicts one by one

$\mathcal{H}$: components of $G[V_1 \cup V_2]$ having conflicting (1-mono) vertices



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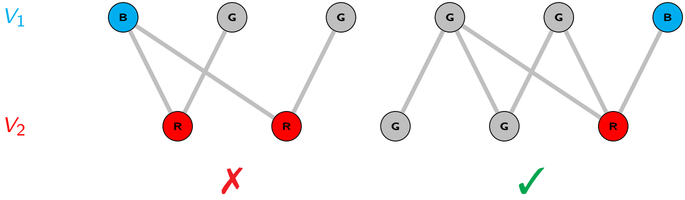

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2. H contains a 1-mono $u \in V_2$ with at least two neighbours in H
3. H contains none of the previous

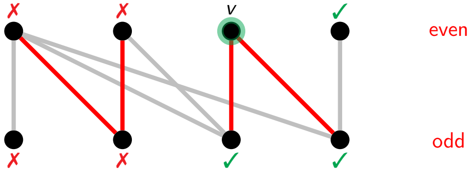
A useful lemma

Lemma

Let $s \in \{2,3\}$, and let H be a connected bipartite graph whose edges are labelled 1 or s. Consider any vertex v in any part $V_i \in \{V_1, V_2\}$ of H. We can relabel the edges of H with 1 and s so that:

- $d_s(u)$ is odd (even, resp.) for every $u \in V_i \setminus \{v\}$, and
- $d_s(u)$ is even (odd, resp.) for every $u \in V_{3-i}$.

Note: Remains true, even if some "contribution" from outside



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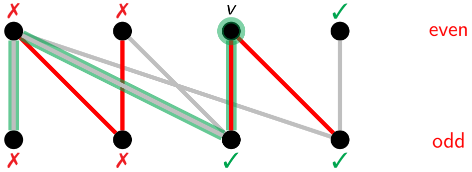

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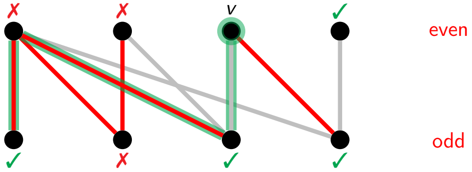

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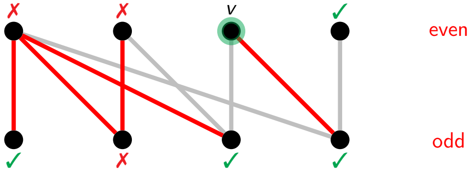

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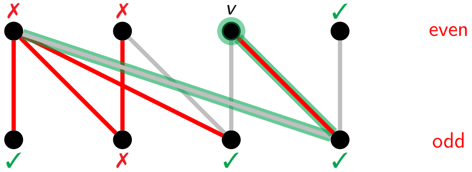

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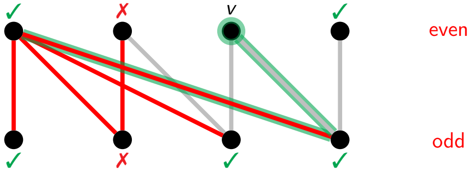

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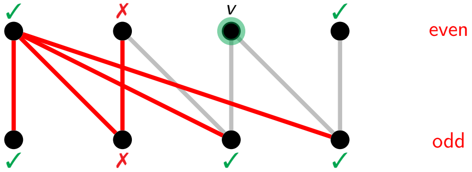

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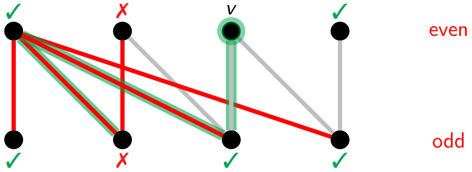

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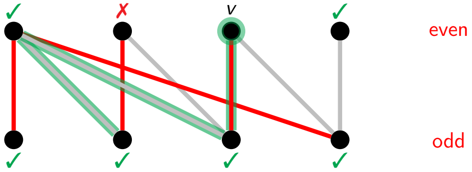

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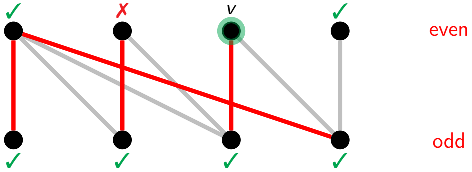

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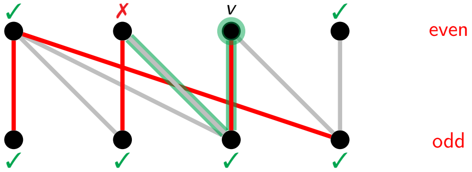

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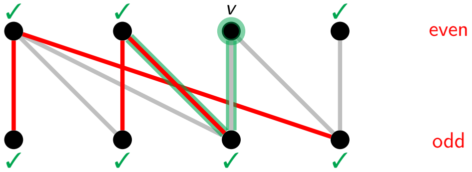

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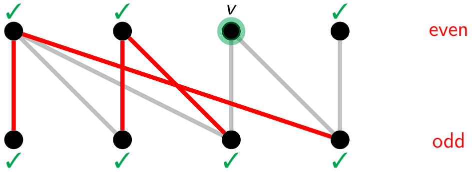
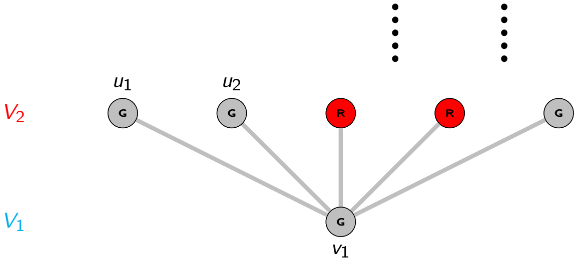


Case 1: H has either 1) a 1-mono $v_1 \in V_1$ with two 1-mono degre-1 neighbours $u_1, u_2 \in V_2$, or 2) a 3-mono $v \in V_1$

First situation:



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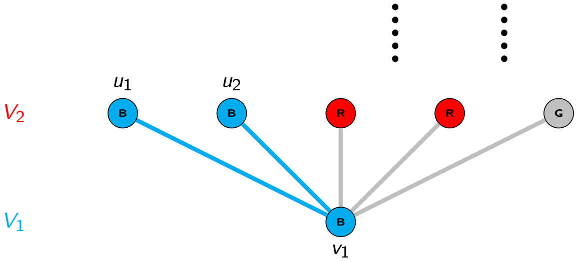

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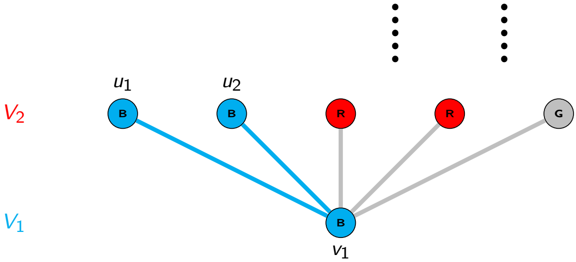

... and then 2nd situation, keeping in mind that the only 3-mono in V_2 are u_1, u_2

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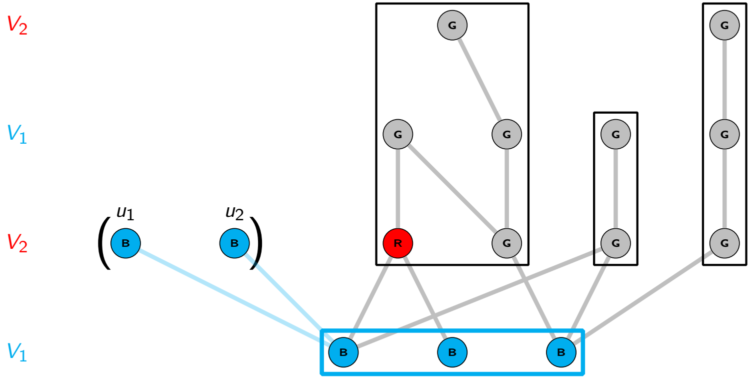
Second situation:



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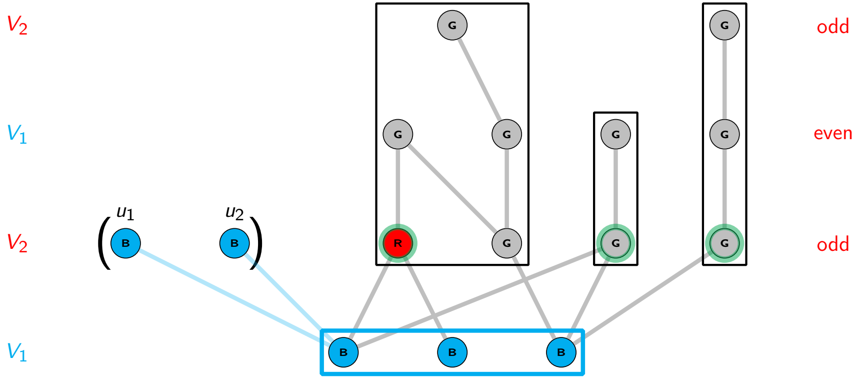

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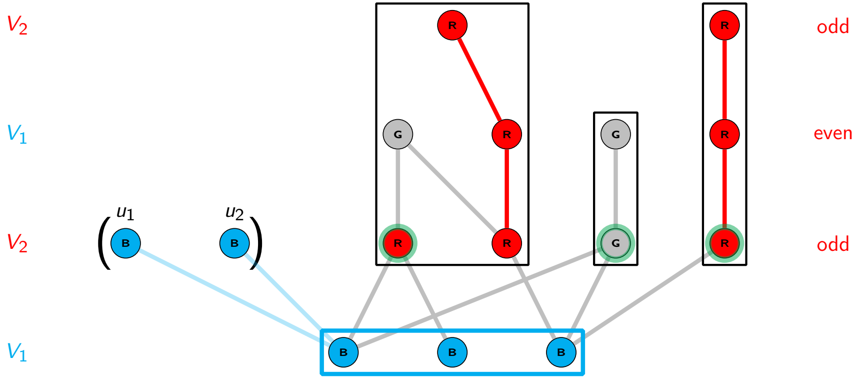

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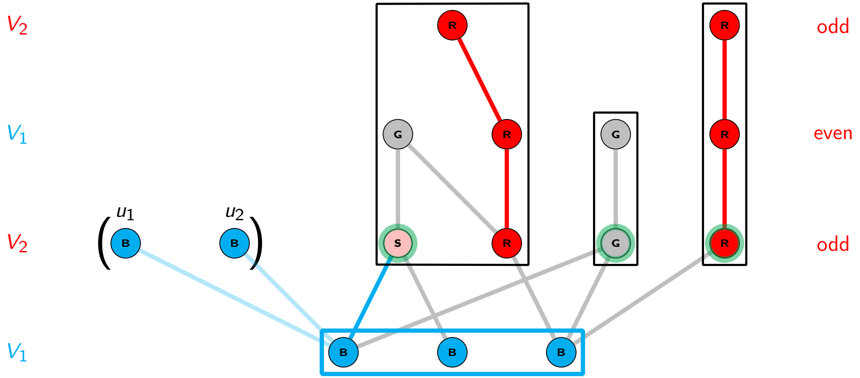

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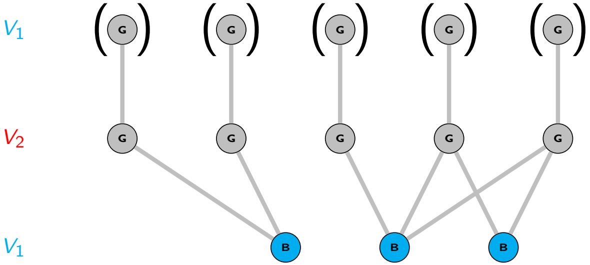

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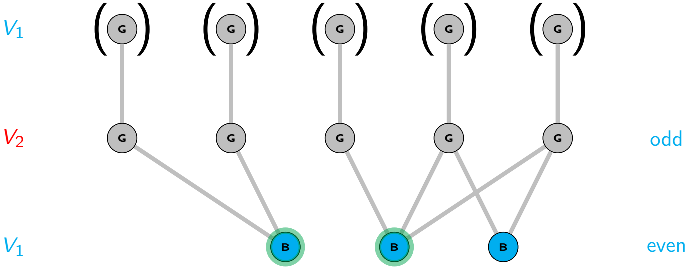

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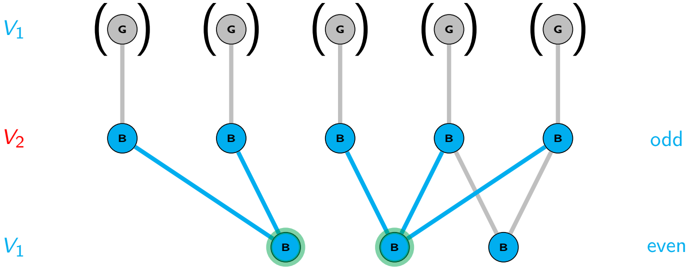

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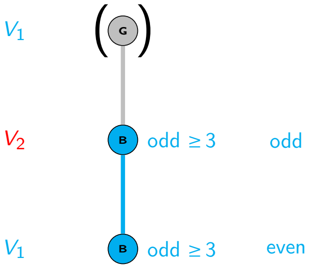

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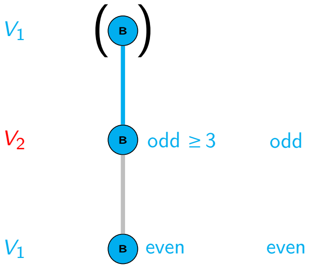

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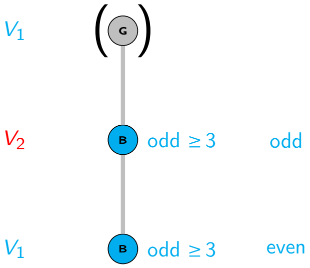

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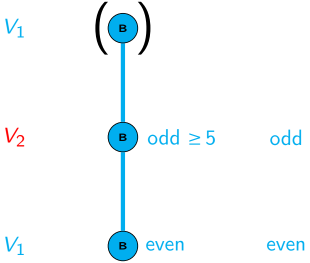
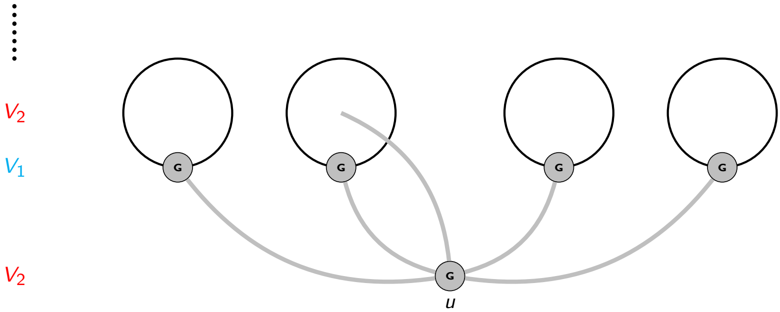
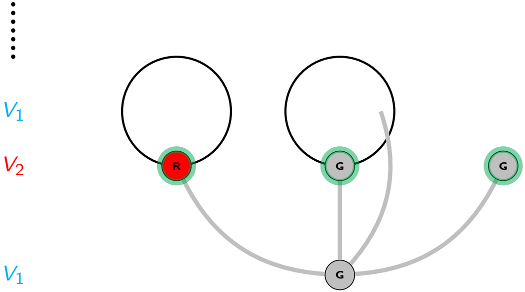


Case 2: H contains a 1-mono $u \in V_2$ with at least two neighbours in H



Looking closer at components of $H - u$



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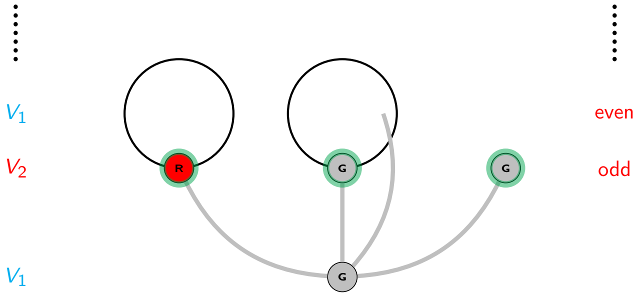
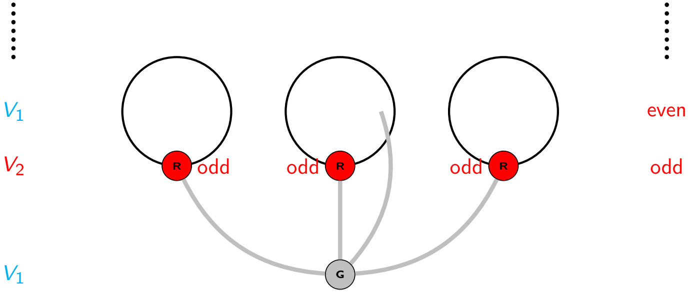


Nice components

Nice component: no conflict, or at least two neighbours with even 2-degree, or only one neighbour with even 2-degree at least 2

$\Rightarrow$ **can make sure no conflict in the component!**



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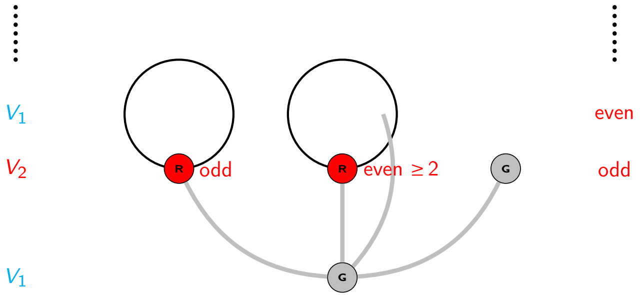

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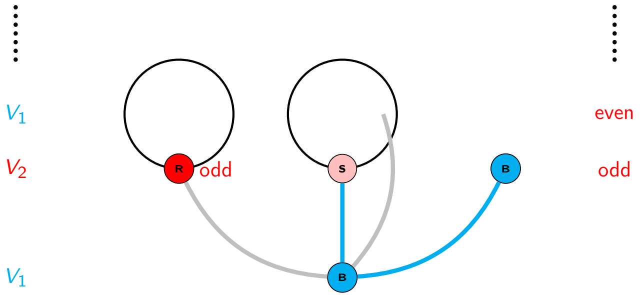
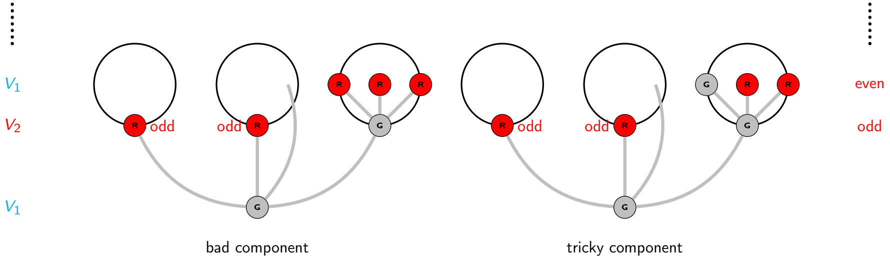
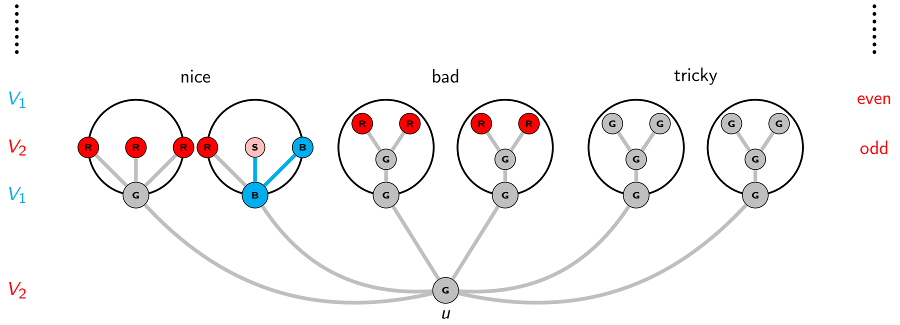


Bad and tricky components

Bad component: exactly one neighbour with even 2-degree, being 1-mono
Tricky component: that 1-mono neighbour is adjacent to a 1-mono neighbour



Global picture



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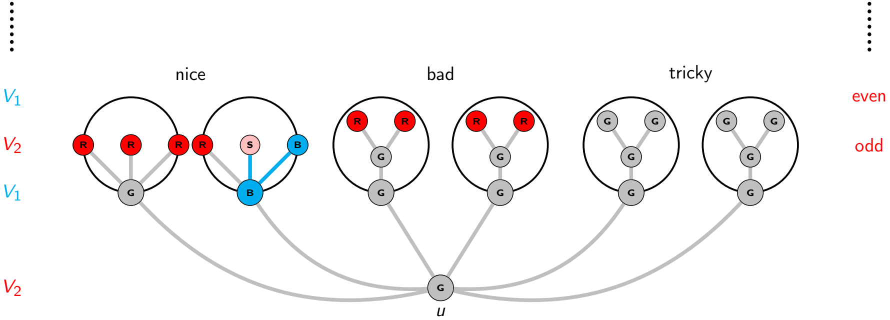
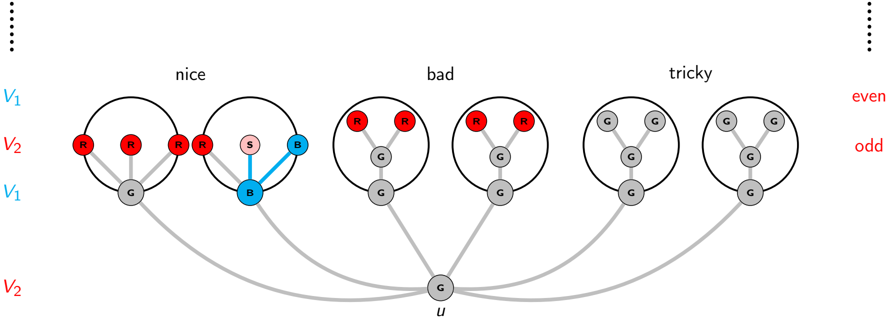


Some terminology:

- N_n: number of nice components

Global picture



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- N_n: number of nice components
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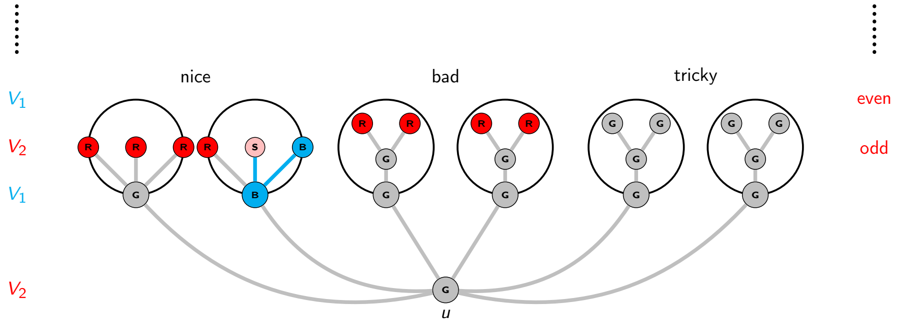

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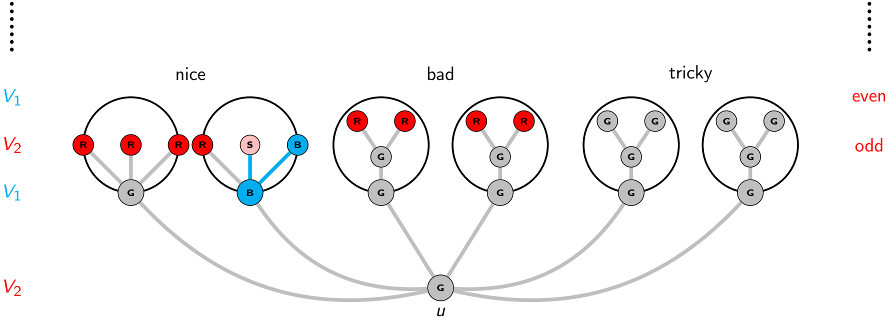

Some terminology:

- N_n: number of nice components
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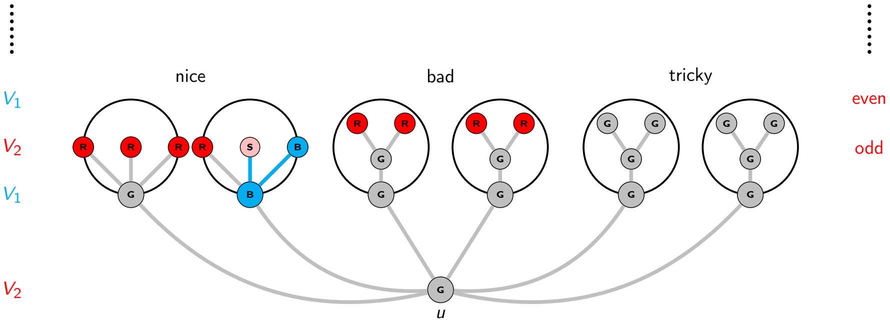
- N_{an}: number of neighbours with 2-degree 0 in nice components ($N_{an} \geq N_n$)

First case: $N_t > 0$



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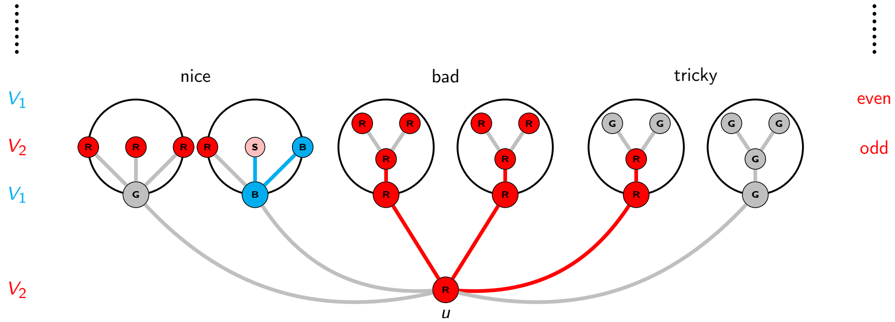

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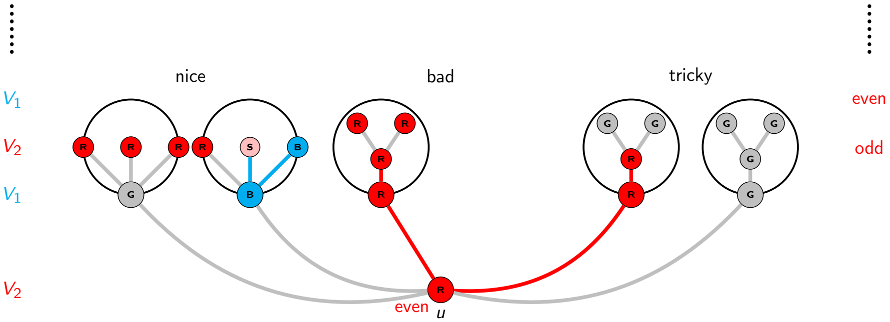

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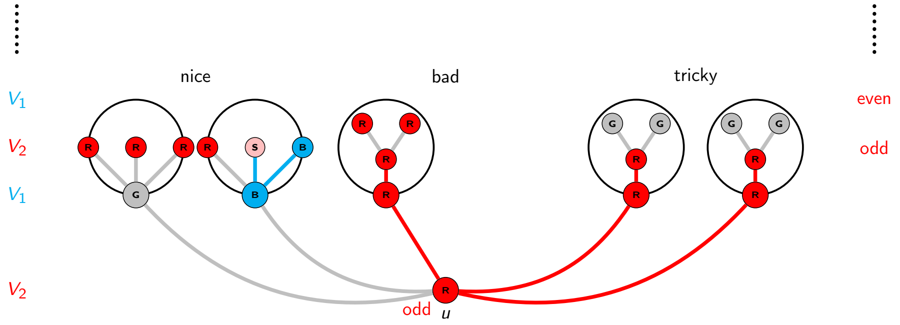

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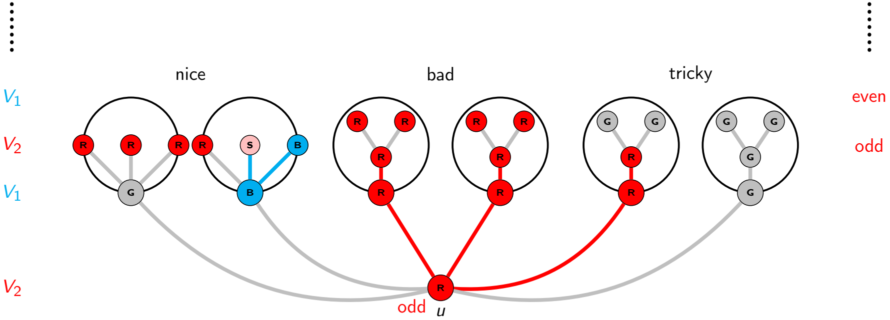

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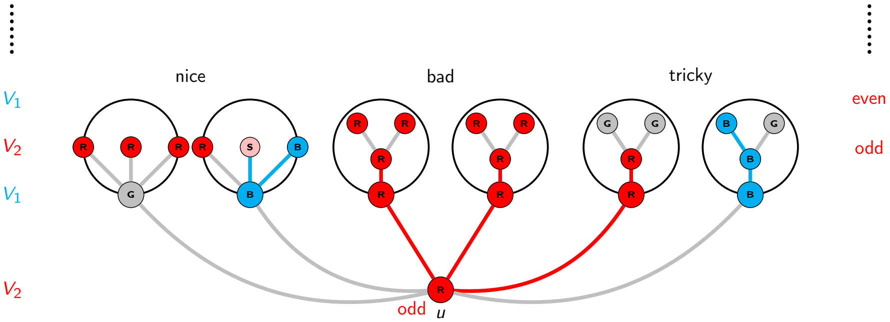
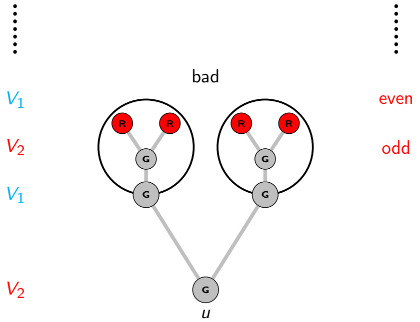


Second case: $N_{an} = N_n = 0$



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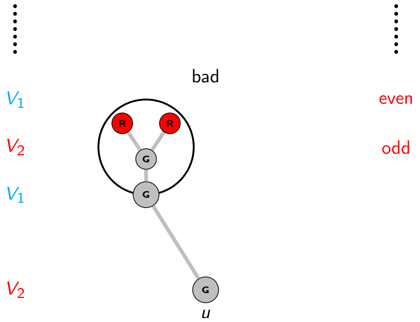

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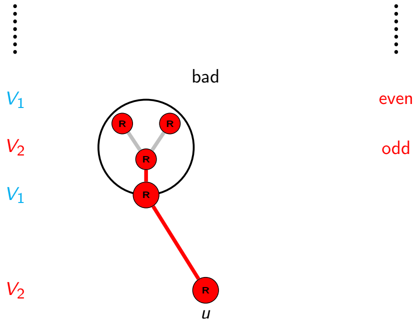

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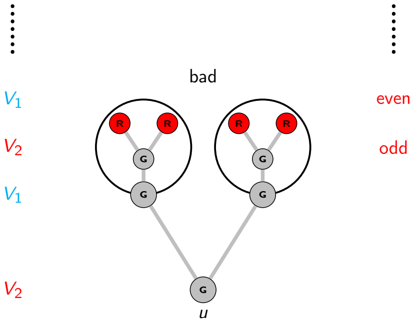

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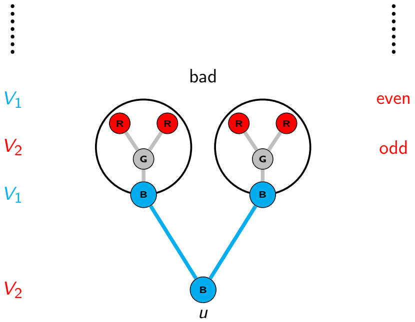

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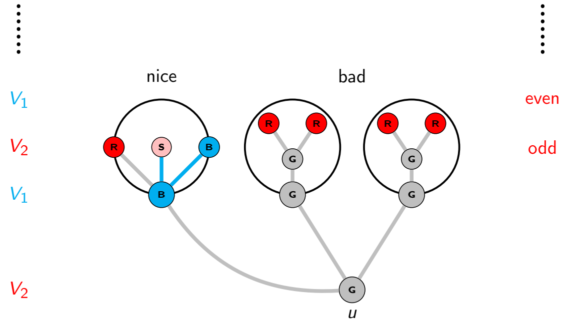

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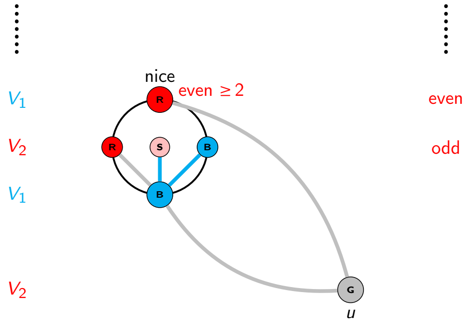

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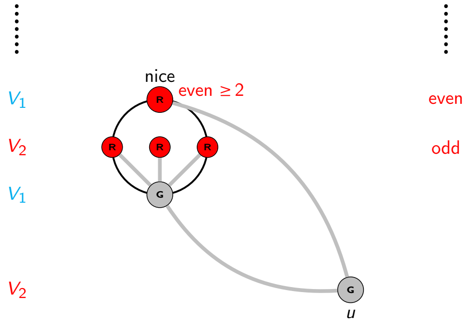

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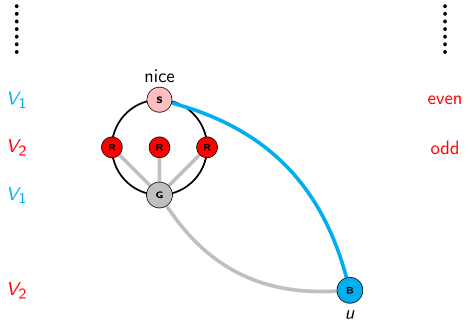

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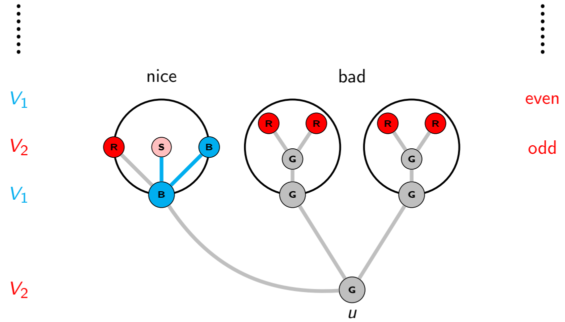

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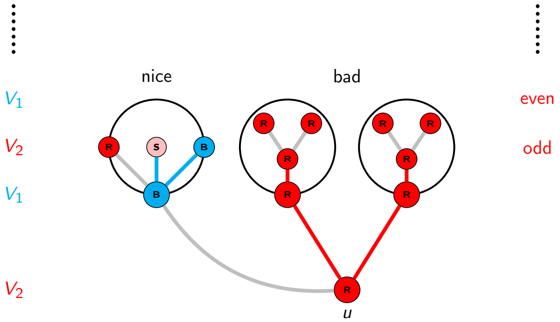

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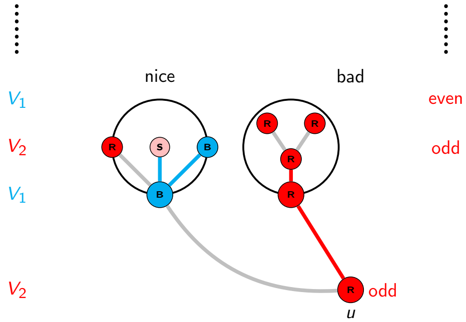

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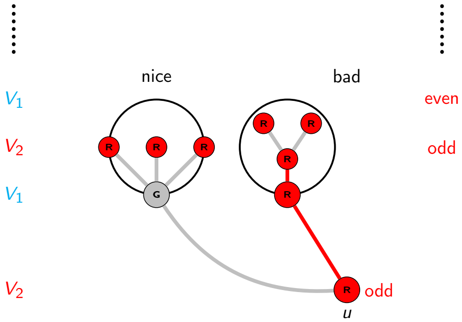

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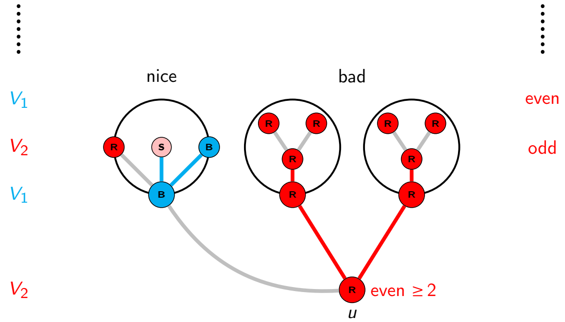

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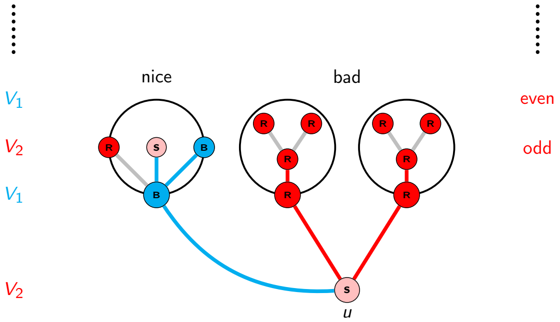

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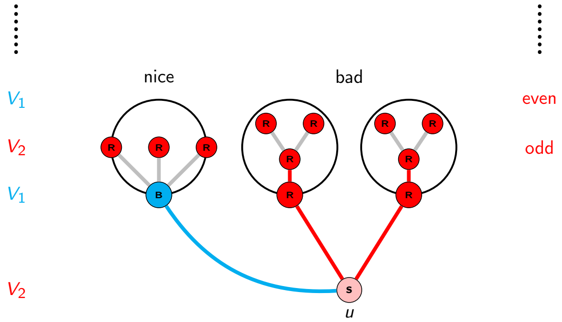
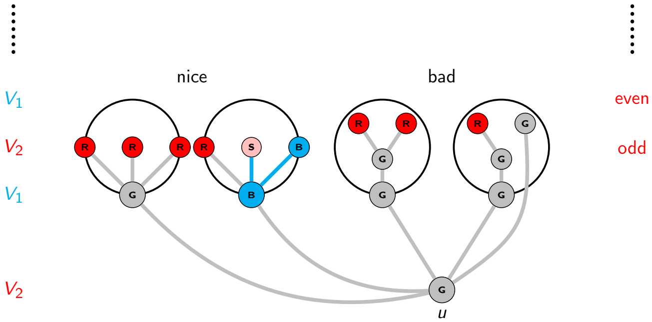


Fourth (and last) case: $N_{an} \geq 2$

$A = \{a_1, \ldots, a_r\}$: neighbours with 2-degree 0 not "main neighbour" in a bad comp.



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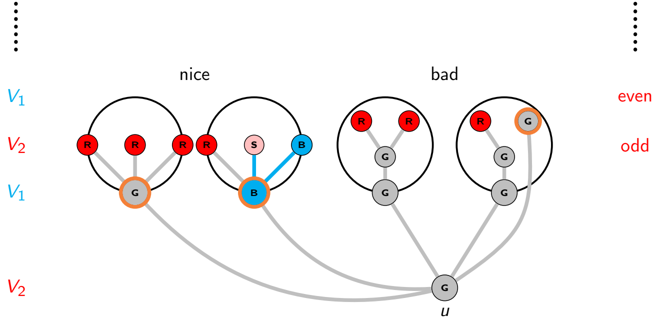

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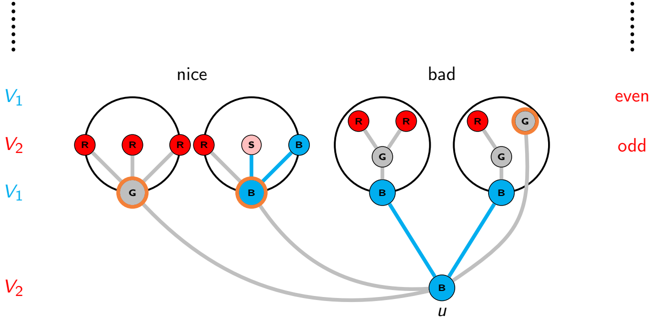

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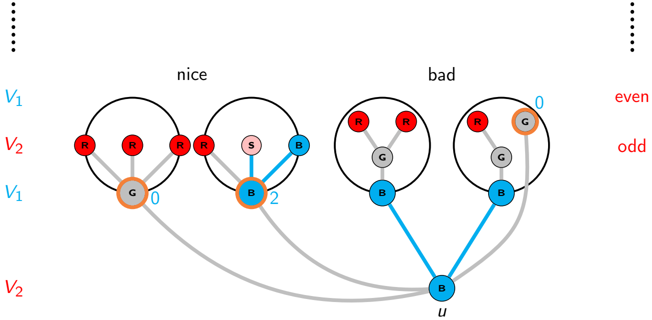

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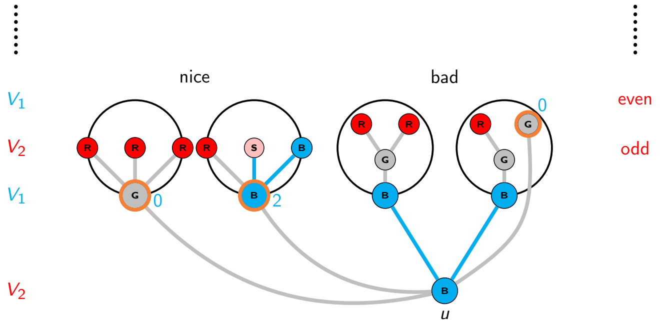

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Goal: Relabel some ua_i's with 3 so that u is not in conflict with the a_i's

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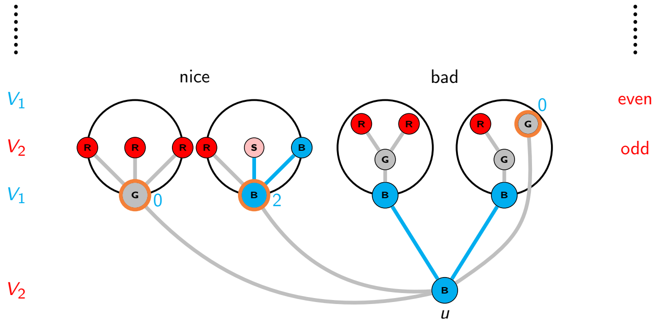

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$\Rightarrow$ possible because $N_{an} \geq 2$

Polynomial representation

- For every $i \in \{1, \ldots, r\}$, let X_i be a variable taking value in $\{0, 1\}$
- $X_i = 0$ means label 1 on ua_i, while $X_i = 1$ means label 3 on ua_i

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$$P(X_1, \ldots, X_r) = \prod_{i=1}^{r} \left(\sum_{\substack{j=1 \\ j \neq i}}^{r} X_i + N_b - n_i \right)$$

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Combinatorial Nullstellensatz (Alon, 1999)

Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1,\ldots,x_n)$ be a polynomial in $\mathbb{F}[x_1,\ldots,x_n]$. Suppose the total degree of f is $\sum_{i=1}^{n} t_i$, where each t_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ is non-zero. If $S_1,\ldots,S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, then there are $s_1 \in S_1, s_2 \in S_2,\ldots,s_n \in S_n$ so that $f(s_1,\ldots,s_n) \neq 0$.

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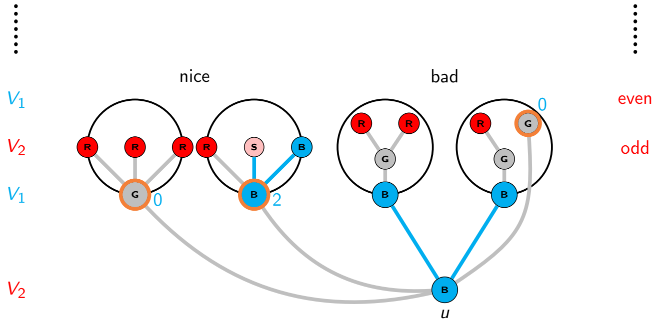
- Here, just consider the monomial $\prod_{i=1}^{r} X_i$ $\Rightarrow$ the desired x_i's exist!

Fourth (and last) case: $N_{an} \geq 2$



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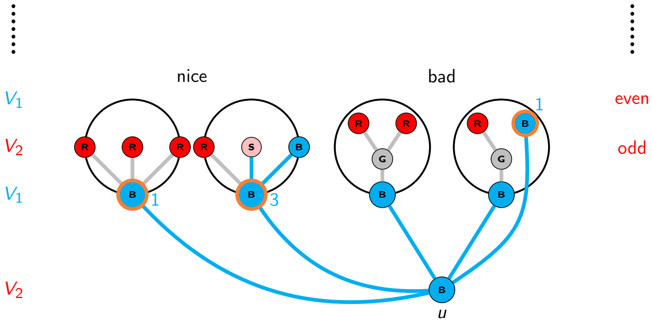

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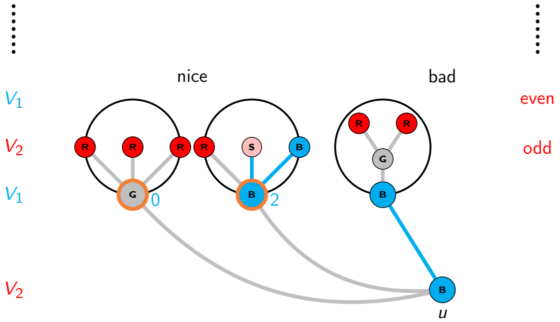

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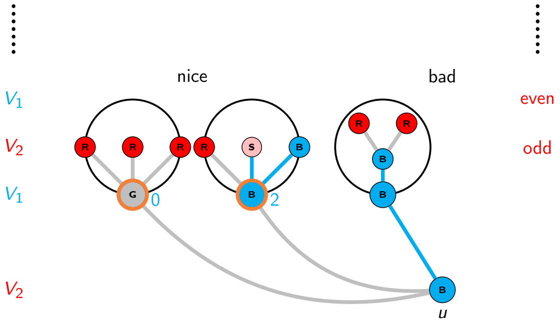
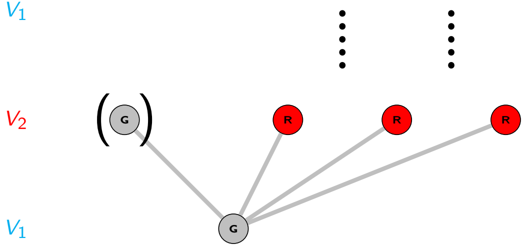


Case 3: None of Cases 1 and 2 applies

1. H contains either:
   - a 1-mono $v_1 \in V_1$ with two 1-mono degre-1 neighbours $u_1, u_2 \in V_2$, or
   - a 3-mono $v \in V_1$
2. H contains a 1-mono $u \in V_2$ with at least two neighbours in H



Case 3: None of Cases 1 and 2 applies

1. H contains either:
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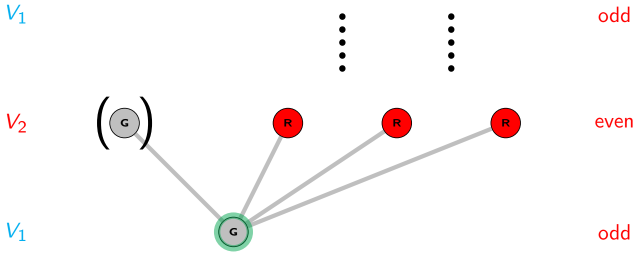

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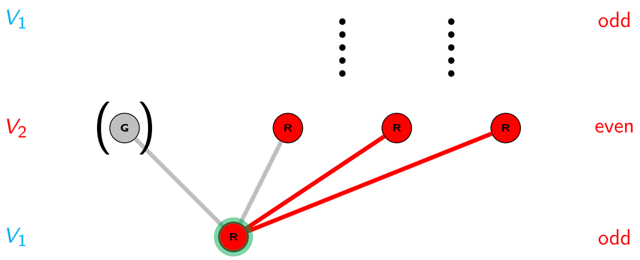

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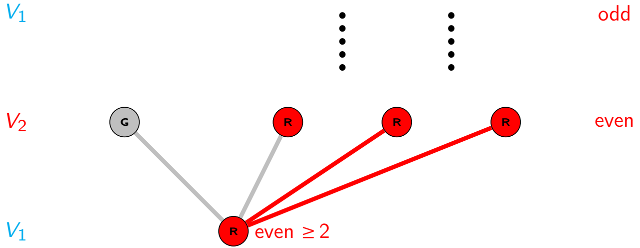

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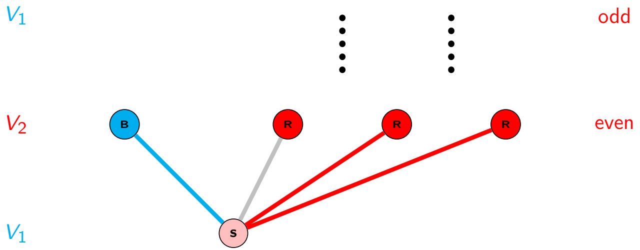

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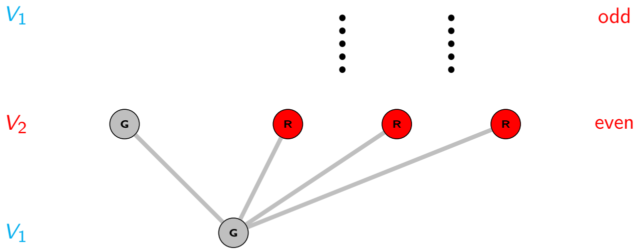

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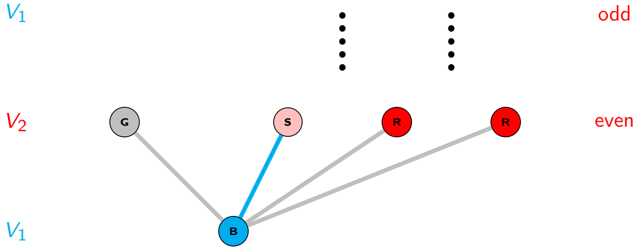

End of the proof, phew...

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Thank you for your attention!