On augmenting matchings via bounded-length augmentations

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Université Nice-Sophia-Antipolis, France

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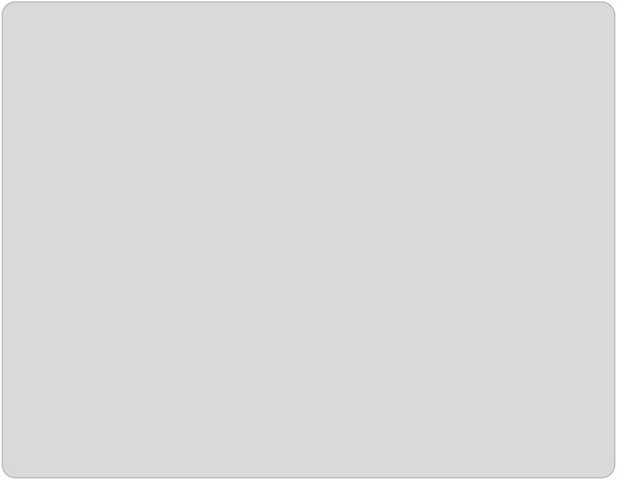
Introduction

Cast



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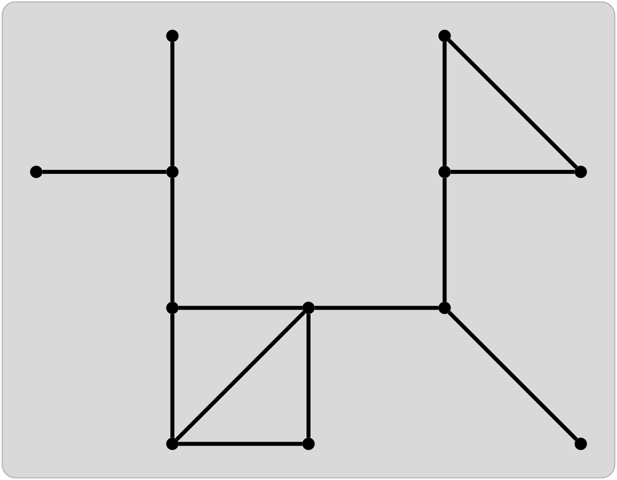
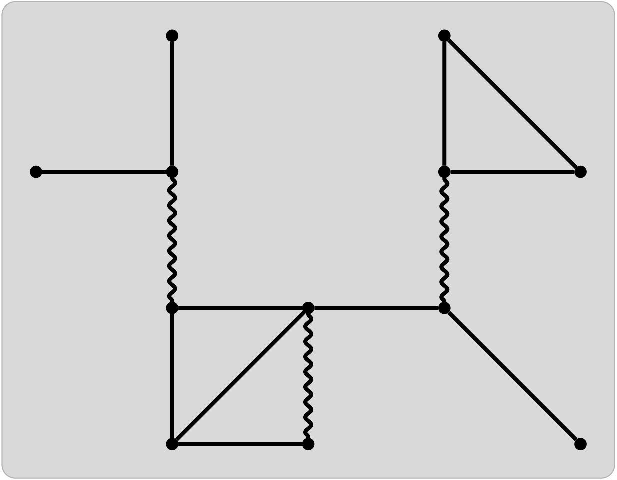
Graph



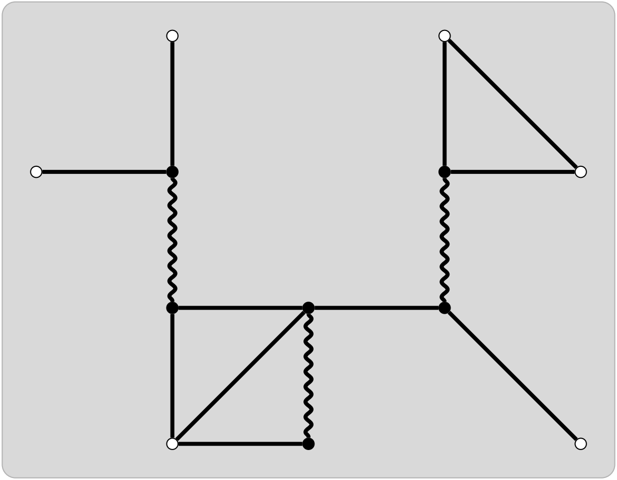
Cast

Graph, Matching



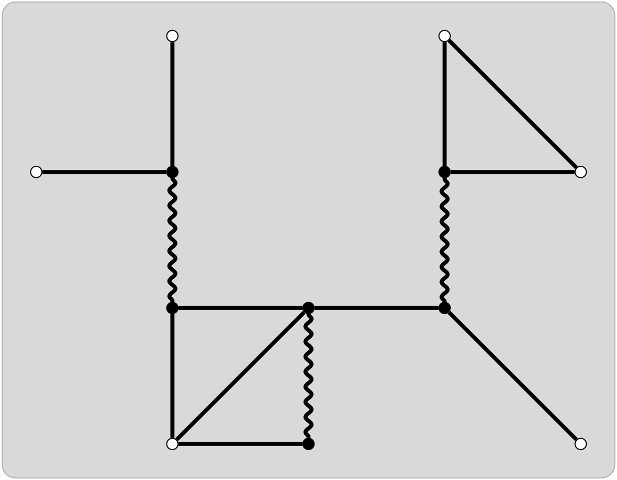
Cast

Graph, Matching
Exposed vertex, Covered vertex



Augmenting a matching

Augmenting path, Augmentation



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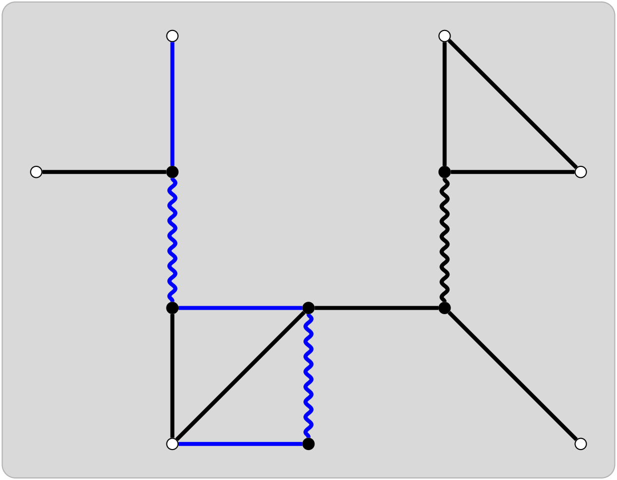

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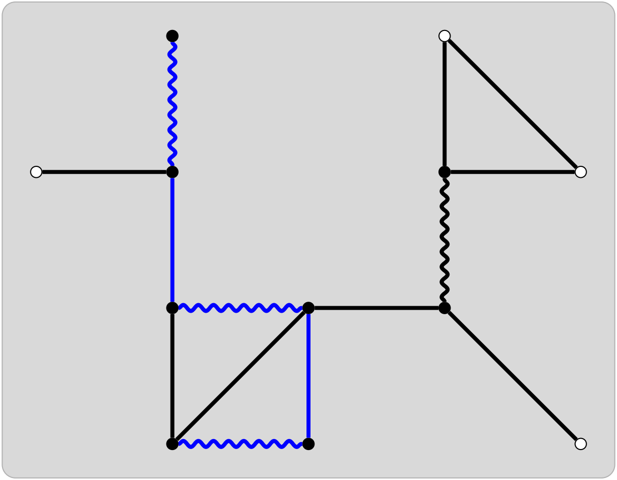

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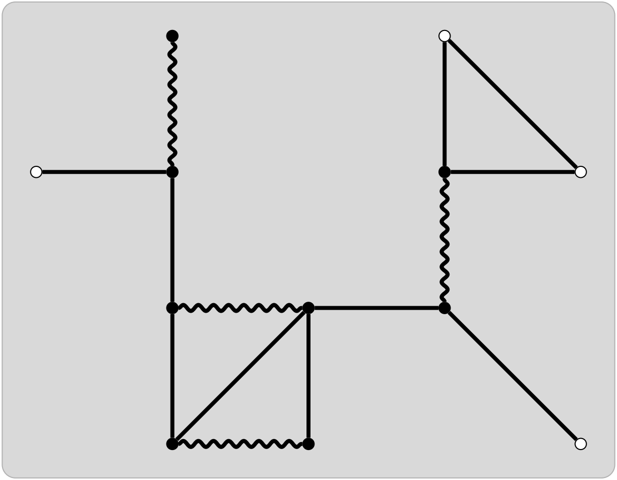

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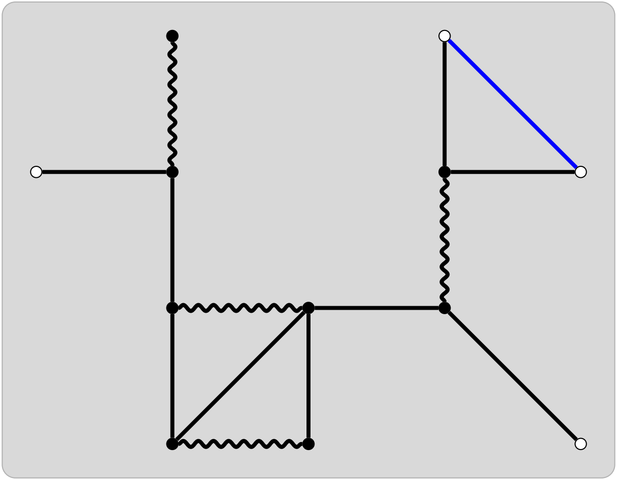

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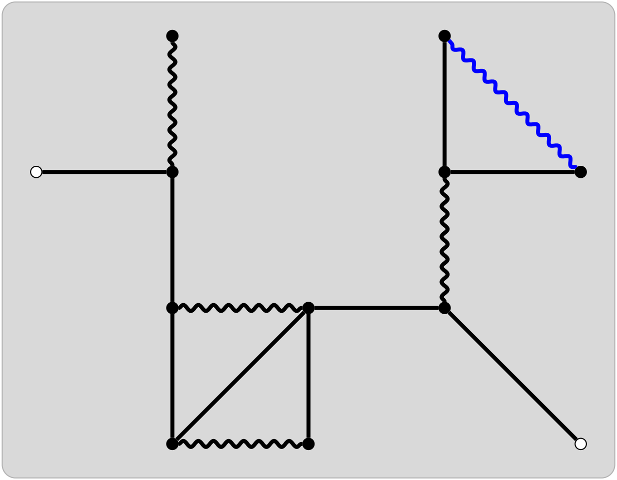

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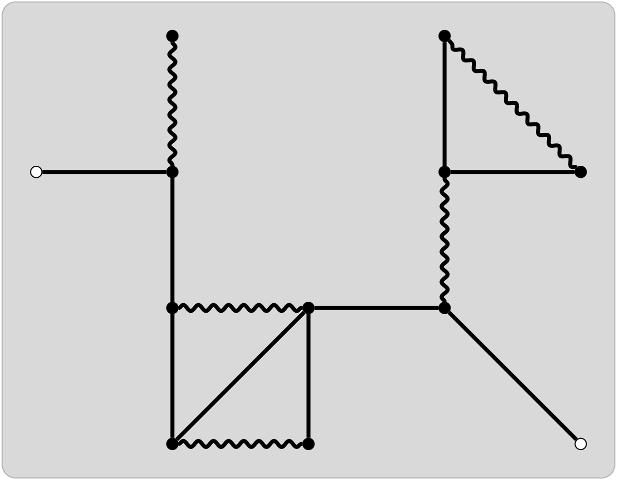

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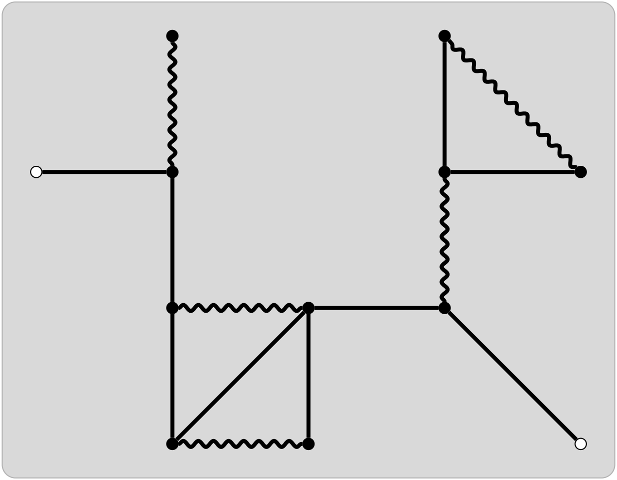

Augmentation ⇒ Bigger matching.

Berge and Edmonds' results

Maximum matching = Biggest matching.
$\mu(G)$ = Cardinality of a maximum matching of G.

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Theorem [Berge, 1957]

Maximum matching $\Leftrightarrow$ No augmenting path.

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Finding augmenting paths?

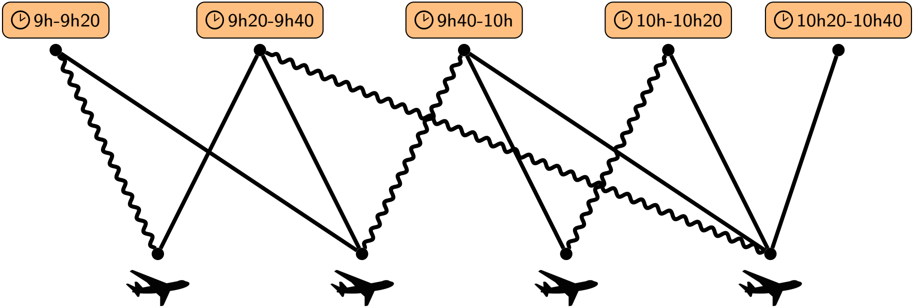
Theorem [Edmonds' Blossom Algorithm, 1965]

Detection in polynomial time.

Hence, $\mu(G)$ can be determined in poly-time.

Today's motivation

Plane → Suitable landing slot times (edges) + Scheduled one (matching).



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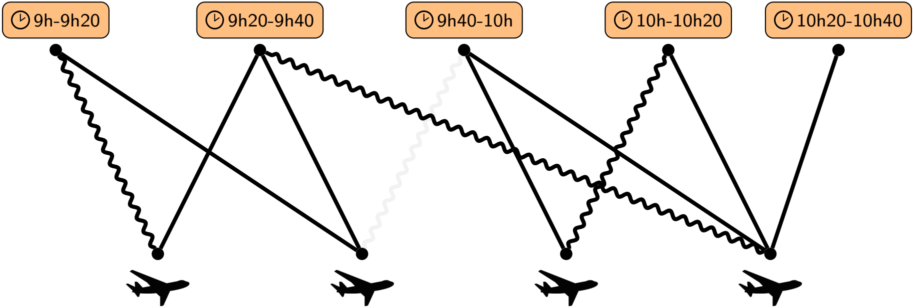
Issue: Plane 2 is delayed...



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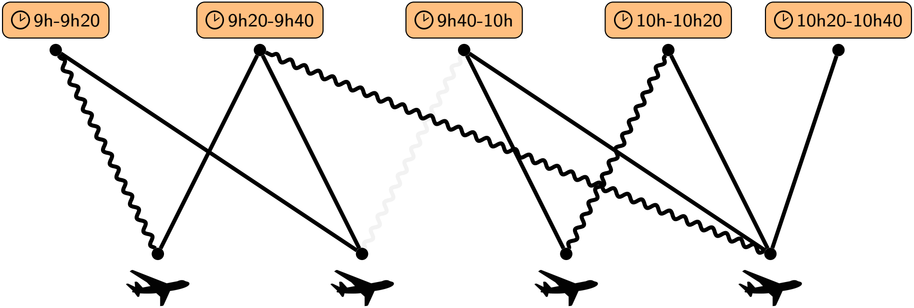
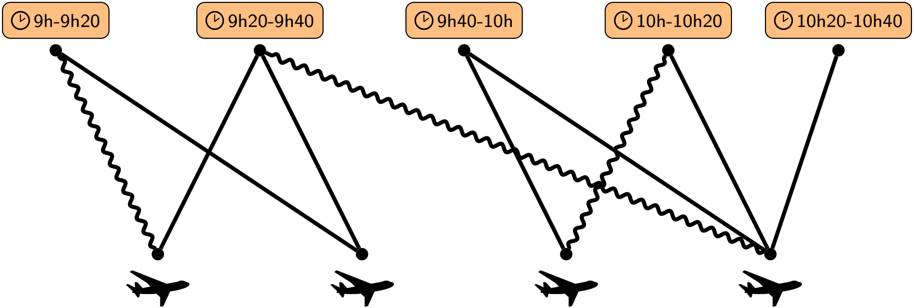


How to fix that??

Motivation

Re-scheduling a lot is not acceptable! ⇒ Cannot start over from scratch.
⇒ Modify the matching "locally", via an augmentation.



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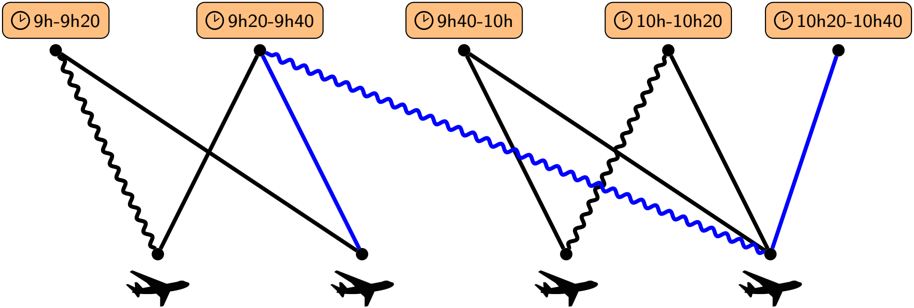

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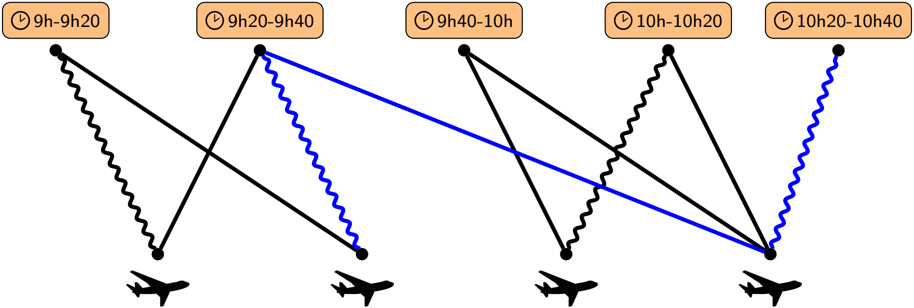

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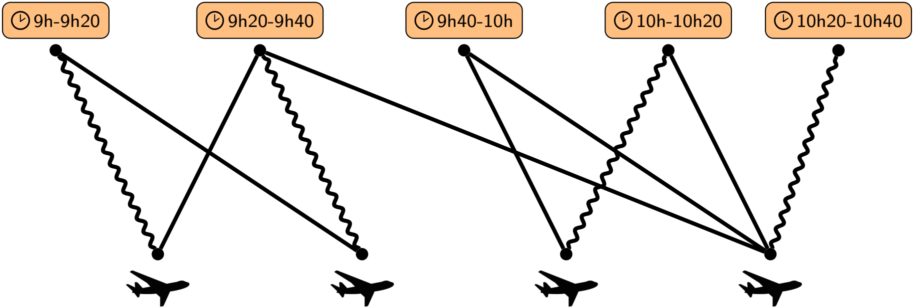

More formally

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1. 1-paths (= pick a free slot), and
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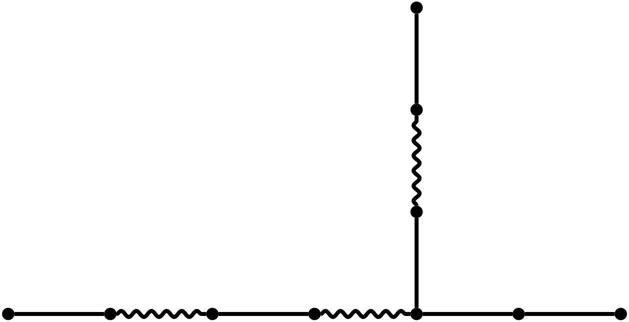
For odd $k \geq 1$, attain a largest matching via ($\leq k$)-augmentations?

$\mu_{\leq k}(G, M)$: Its cardinality for G equipped with M.

Note: $\mu_{\leq 1}(G, \emptyset) = \mu(G)$.

Note: order matters

$k = 5$. First attempt.



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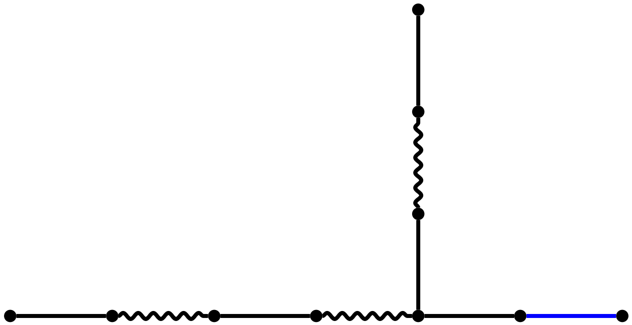

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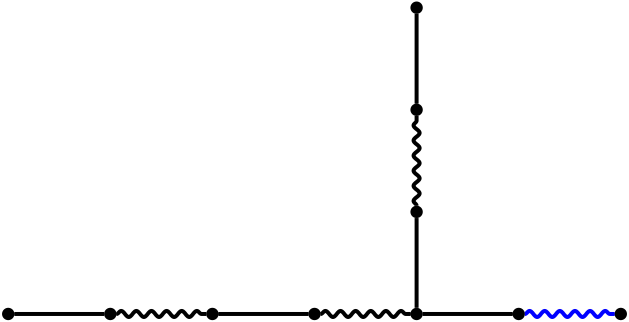

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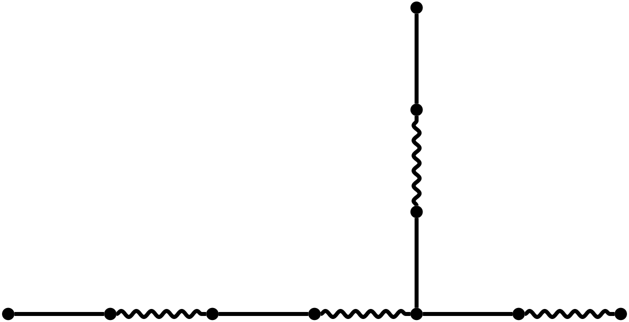

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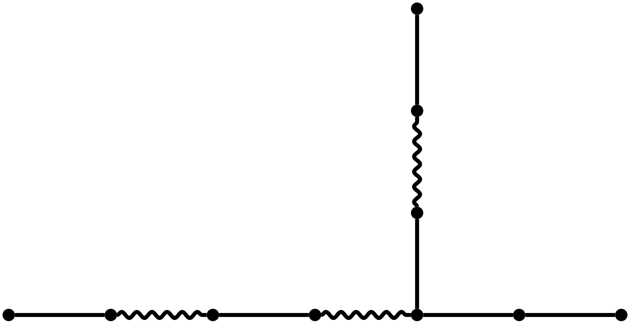

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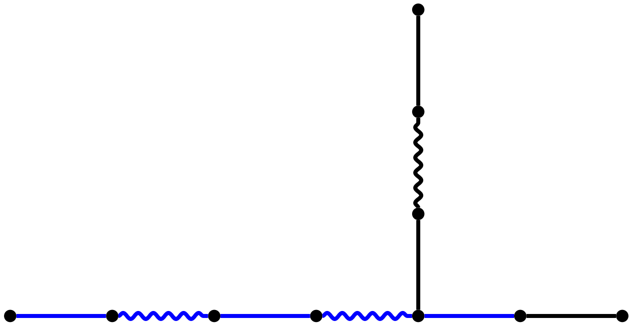

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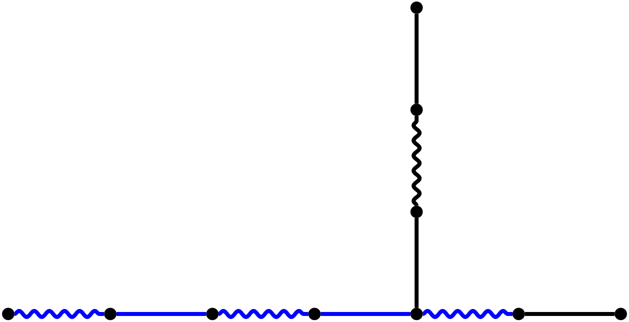

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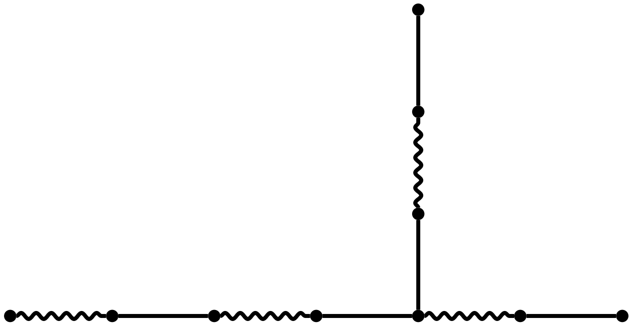

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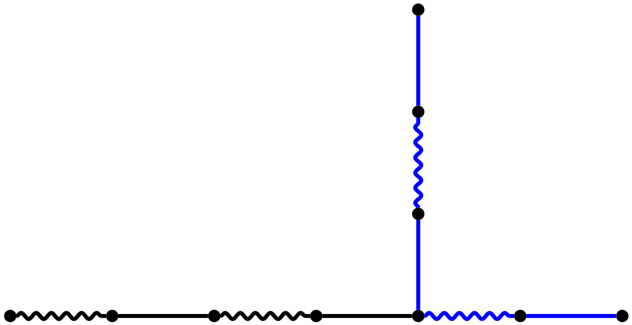

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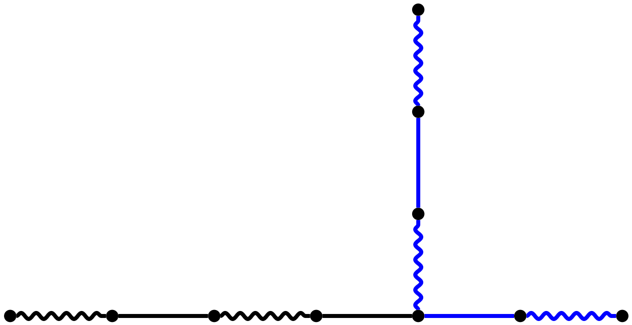

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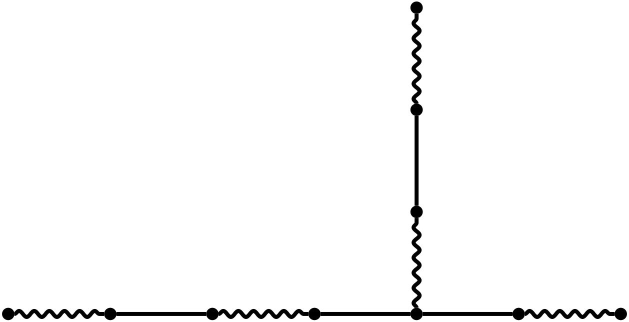

First dichotomy

($\leq k$)-MATCHING PROBLEM – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
Question: What is the value of $\mu_{\leq k}(G, M)$?

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($\leq k$)-Matching Problem – ($\leq k$)-MP
Input: A graph G, and a matching M of G.
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Dichotomy on k:

Theorem [Nisse, Salch, Weber, 2015+]

($\leq k$)-MP is

- in P for $k = 1, 3$;
- NP-hard for every odd $k \geq 5$.

Latter statement true for planar bipartite graphs with $\Delta \leq 3$ and arb. large girth.

Towards a second dichotomy

Summary:

- For $k = 1, 3$, the problem is settled.
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Complexity of $(\leq k)$-MP for trees?

Today's talk:

- $(\leq k)$-MP is in P for caterpillars, subdivided stars, "sparse trees", etc.
- A modified version is NP-complete for trees.

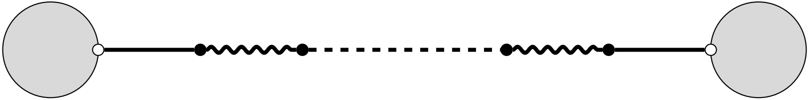
Positive results

Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

$(\leq k)$-MP is in P for paths.

1st key idea: Consider exposed vertices joined **only once** by an augmenting path.



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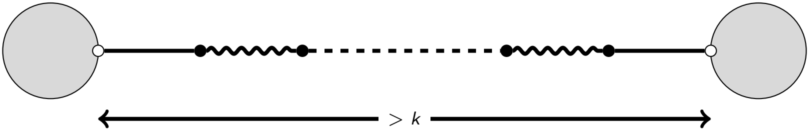

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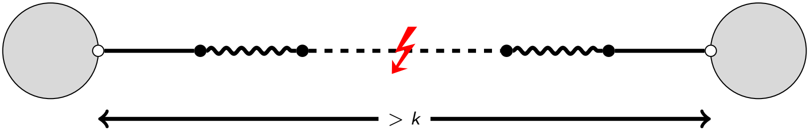

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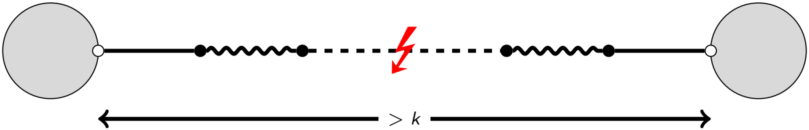

$\Rightarrow$ Decompose the problem into two sub-problems.
In a path $\Rightarrow$ Exposed vertices have one on the left/right at distance $\leq k$.

Easy case: paths

Theorem [Nisse, Salch, Weber, 2015+]

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2nd key idea: We can augment paths joining "consecutive" exposed vertices only.

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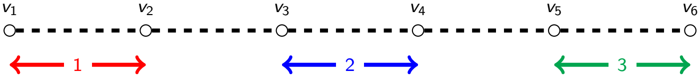
3 $\Rightarrow$ The paths $v_1 \ldots v_2$, $v_3 \ldots v_4$ and $v_5 \ldots v_6$ have length $\leq k$ and alternate. So



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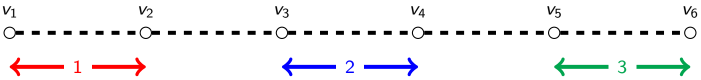


yield the same matching.

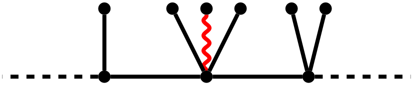
⇒ In a path, just go from left to right, and augment paths when possible. ∎

Caterpillars

Theorem [B., Garnero, Nisse, 2017+]

($\leq k$)-MP is in P for caterpillars.

Remark: Matched leaf edge $\Rightarrow$ Simplification.



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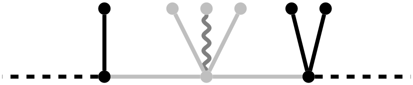

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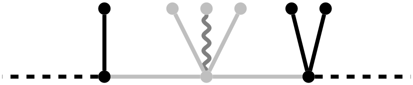

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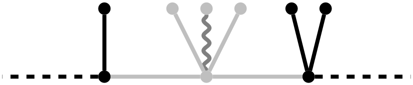

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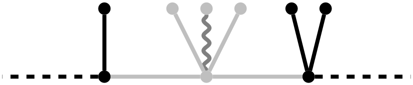

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Focus on caterpillars with $\Delta = 3$ ($\sim$ paths).

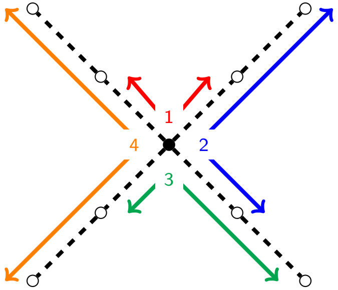
Essentially, again go from left to right. ∎

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for subdivided stars.

Claim: Augmentations through the root should behave in a path way:



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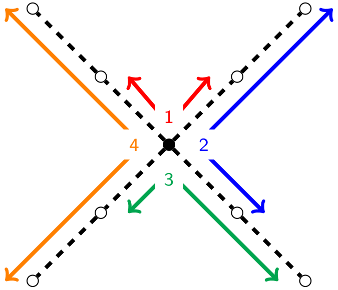

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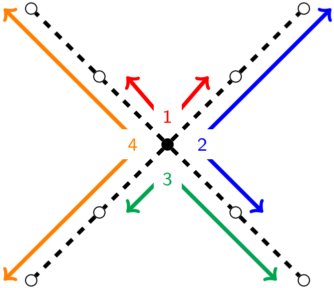

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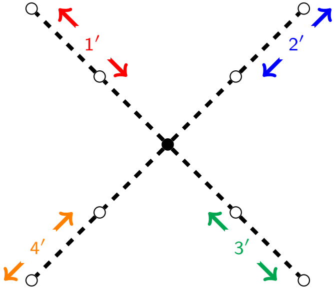

(because 1, 2, 3 and 4 are augmenting $(\leq k)$-paths.)

Subdivided stars

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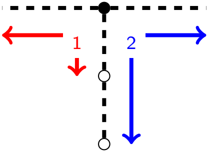
For an inner-branch, an equivalent augmentation can be performed



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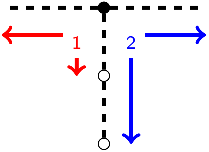

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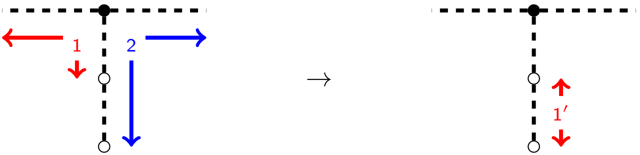

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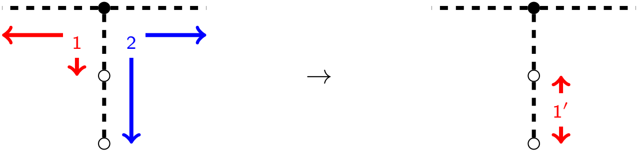

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Augmentations through the root → Alter the parity of the end-branches only.

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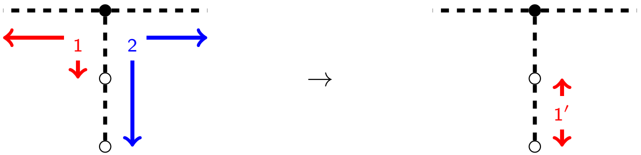

... and it retains the parity of the number of exposed vertices along that branch.

Augmentations through the root → Alter the parity of the end-branches only.

Along a branch with α exp. vertices, $\lfloor \alpha/2 \rfloor$ augmentations can be done (path):

- α even: all exposed can be matched along.
- otherwise: all but one.

Subdivided stars

Theorem [B., Garnero, Nisse, 2017+]

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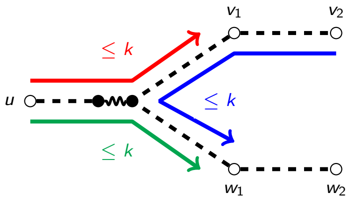

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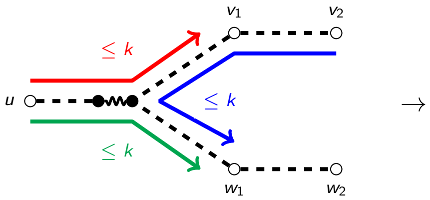

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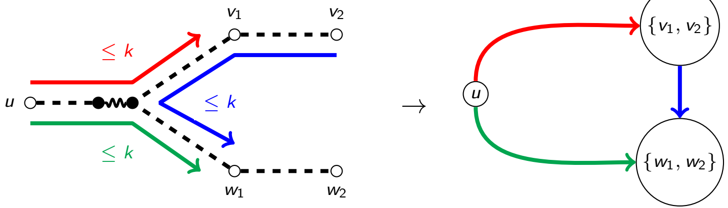

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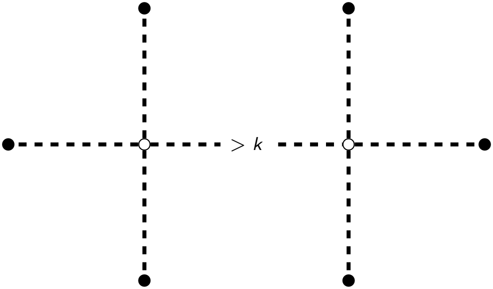
To summarize:

1. If necessary, do an augmentation involving the root.
2. If possible, join two odd branches via root augmentations.
3. Finally, match the remaining exposed vertices on the branches.

$\Rightarrow$ Polynomial-time algorithm. ■

Going to sparse trees

k-**sparse tree:** Vertices with degree ≥ 3 are at distance $> k$.

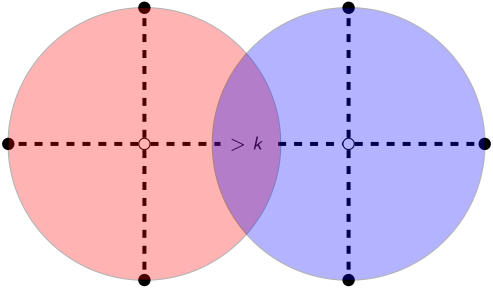


$(\leq k)$-MP for k-sparse trees

Theorem [B., Garnero, Nisse, 2017+]

$(\leq k)$-MP is in P for k-sparse trees.

Idea: Consider subdivided stars, and build a solution from bottom to top. ■



Negative results

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NP-hardness proof: Need some forcing mechanisms.

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Good news: Some properties of $(\leq k)$-MP derive to $(= k)$-MP:

- NP-hardness for odd $k \geq 5$;
- all polynomial-time algorithms for classes of trees.

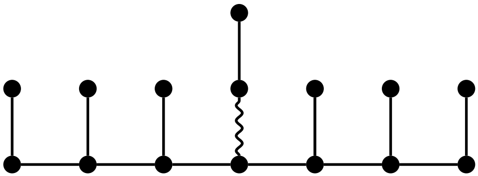
On (≤ 3)-MP and $(= 3)$-MP

Recall that (≤ 3)-MP is in P.

Theorem [B., Garnero, Nisse, 2017+]

$(= 3)$-MP is NP-hard.

Proof: Reduction from 3-SAT. Just need variable gadget:



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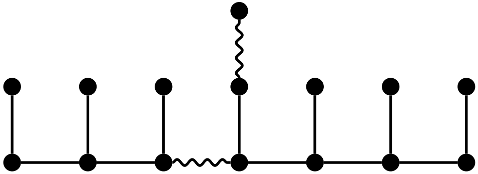

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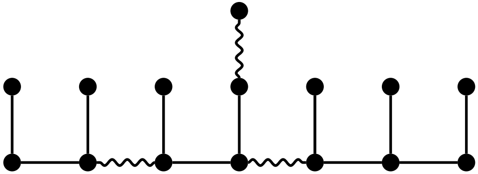

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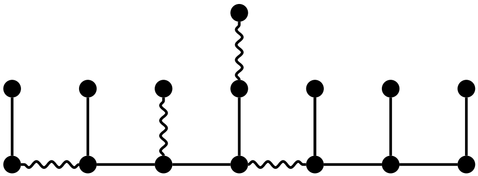

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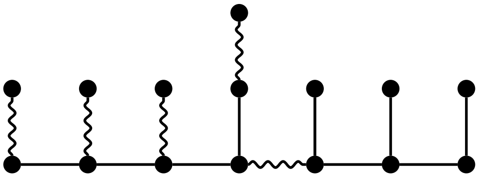

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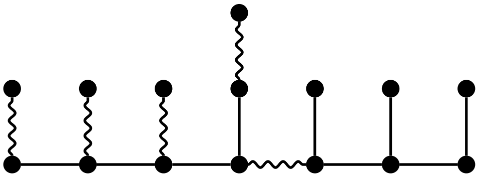

Longest sequence: Matched edges on all spikes of a single side.

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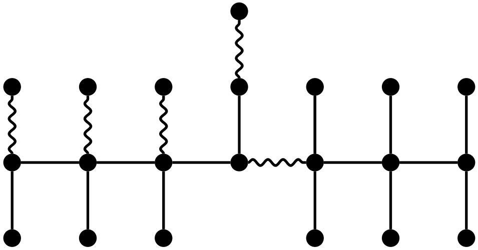
Attach a leaf to the base of every spike. Previous remark still applies.



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Add a variable gadget G_i for each x_i. Pushing left=True. Pushing right=False.

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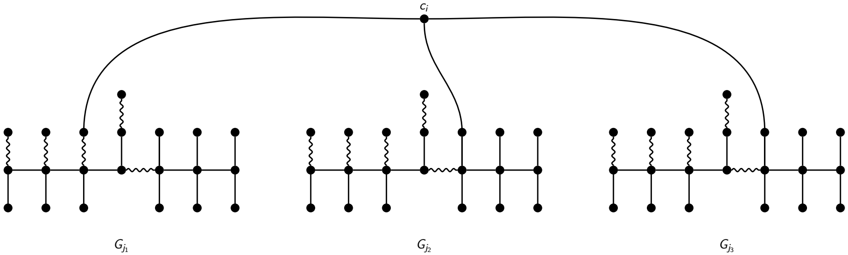
Add a variable gadget G_i for each x_i. Pushing left=True. Pushing right=False.

Next add a clause vertex c_i for every clause C_i, and, for every distinct literal ℓ_j it contains, join c_i and one non-used spike of G_i (left if positive, right otherwise).



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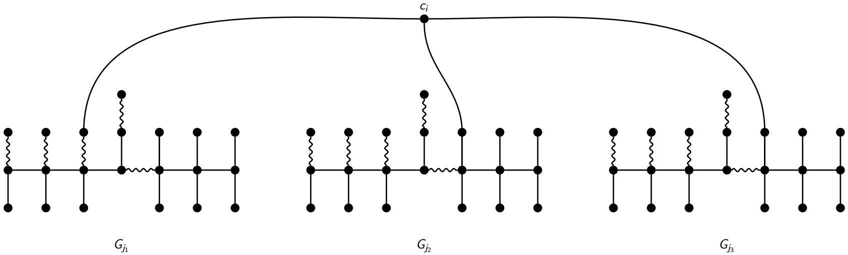

$\Rightarrow$ One additional augmentation covering c_i can be done.

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Maximum $\#$ of 3-augmentations:

1. For every G_i, push the matching to the left (x_i true) or to the right (x_i false).
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$\Rightarrow$ Maximum $\mu_{=3}$ achievable is

$$(\#\text{variables} \cdot \#\text{spikes}) + \#\text{clauses},$$

which is attainable iff F is satisfiable. ■

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We have $\Delta \leq 4$ in the reduction.

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$+$ by slight modifications, we can also guarantee $\Delta \leq 3$.

$(= k)$-MP in trees for non-fixed k

Modified version:

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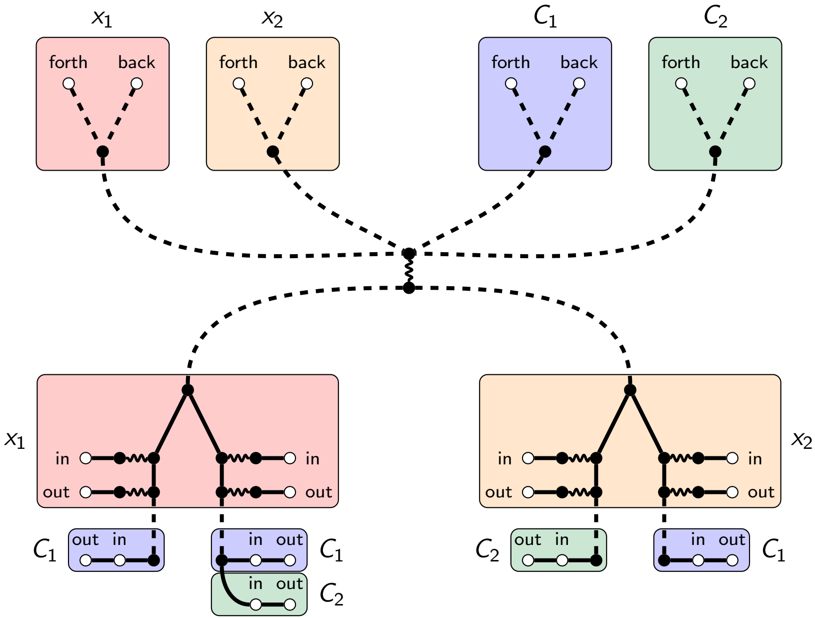
Negative result for trees:

Theorem [B., Garnero, Nisse, 2017+]

$(=)$-MP is NP-hard for trees.

Proof (sketch): Reduction from 3-SAT.

$(=)$-MP in trees



x_1
forth
back
x_2
forth
back
C_1
forth
back
C_2
forth
back
x_1
in
out
in
out
C_1
out in
in out
C_1
in out
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x_2
in
out
in
out
C_2
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Lengths of the dashed paths chosen so that:

- for each x_i, open either the *true* or *false* gate;
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⇒ Needed k depends on #clauses and #variables. ■

Conclusion

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Thank you for your attention!