Edge-partitioning a graph into paths: beyond the Barát-Thomassen conjecture

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T: tree with $|E(T)|$ dividing $|E(G)|$.

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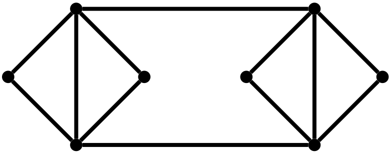

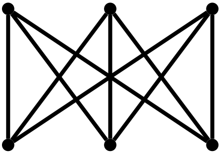

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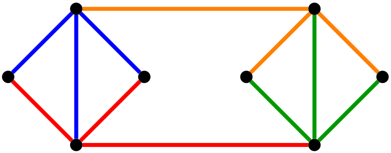
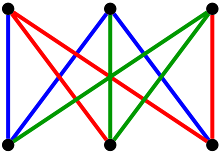


S_4-decomposition



P_3-decomposition

The Barát-Thomassen conjecture

Divisibility condition is understood throughout.

Conjecture [Barát, Thomassen – 2006]

For every fixed tree T, there exists a positive constant c_T such that every c_T-edge-connected graph admits a T-decomposition.

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Verified for T being:

- a star [Thomassen – 2012],
- a bistar of the form $S_{k,k+1}$ [Thomassen – 2014],
- the tree with degree sequence $(1, 1, 1, 2, 3)$ [Barát, Gerbner – 2014],
- of **diameter at most 4 [Merker – 2015+]**,
- among some family of trees with diameter 5 [Merker – 2015+],

and...

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and...

- the path of length 3 [Thomassen – 2008],
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- a **path of length 2^k [Thomassen – 2014]**,
- the path of length 5 [Botler, Mota, Oshiro, Wakabayashi – 2015],
- **any path [Botler, Mota, Oshiro, Wakabayashi – 2015+]**!

About the result for paths

Theorem [Botler, Mota, Oshiro, Wakabayashi – 2015+]

The Barát-Thomassen conjecture is true for T being any path.

About the proof:

- Generalization of a proof for P_5.

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Our goal: give a somewhat simpler proof with *reasonable* technicalities.

A stronger result

'Stronger' $=$ **degree** is more important than **edge-connectivity**.

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Theorem [B., Harutyunyan, Thomassé – 2015+]

For every $\ell \geq 1$, every 64-edge-connected graph admits a P_ℓ-decomposition provided its minimum degree is large enough.

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More general question

2-edge-connectivity $+$ large minimum degree $\Rightarrow$ path-decomposition???

Overview of the proof

Main ideas:

1. See G as a 'convenient' system H of induced paths.
2. Remove some P_ℓ's from G so that H is eulerian.
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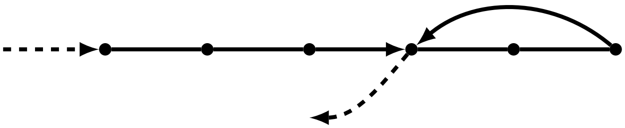

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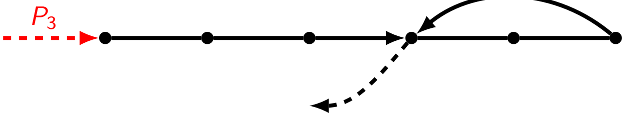

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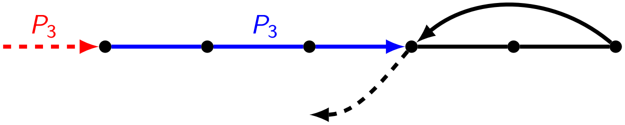

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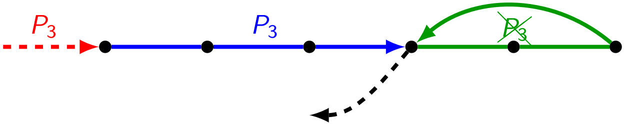

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- **Part 2**: *sparse* and *dense* subgraphs.
- **Part 3**: constructing *systems of edge-disjoint paths*.
- **Part 4**: obtaining such systems with a *tree structure*.

Part 1: Introduction to the problem

Part 2: Fractions of graphs

Part 3: Path-graphs

Part 4: Constructing path-trees

Part 5: Using everything together

Part 6: Conclusion

Sparse and dense subgraphs

α: real number in $[0, 1]$.

Definitions: α-*sparse*, α-*dense*, α-*fraction*

Let H be a spanning subgraph of G. We say that H is α-*sparse* (resp. α-*dense*) if $d_H(v) \leq \alpha d_G(v)$ (resp. $d_H(v) \geq \alpha d_G(v)$) for every $v \in V(G)$. We say that H is an α-*fraction* of G if H is both α-sparse and α-dense.

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k-edge-connectivity + large degree $\Rightarrow$ $1/k$-sparse spanning tree.

Theorem [Ellingham, Nam, Voss – 2002]

Every k-edge-connected graph admits a $1/k$-sparse spanning tree.

(with error term $+2$)

On fractions of graphs

Proposition

Every graph G has a $1/2$-fraction (with error term ± 1).

Proof: If G has an even cycle C, remove the edges of C, apply induction and add the edges of a perfect matching of C to the solution. Otherwise, G is either an odd cycle (in which case the conclusion follows), or has a cutvertex z incident to an 'endblock' B which is either an edge or an odd cycle. Then contract B to z, apply induction, and extend the solution by conveniently choosing some edges of B. ■

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Corollary

If α has a finite binary extension, then G has an α-fraction.

(with constant additive error term)

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Path-graphs

Definition: *path-graph*

A *path-graph* H on $G = (V, E)$ is a couple $(V, \mathcal{P})$ where $\mathcal{P}$ is a set of edge-disjoint paths of G. For every $v \in V$, we define $\mathcal{P}_v$ as the set of paths of $\mathcal{P}$ having v as an endvertex. To H, we associate the (multi)graph H^* on vertex set V and edge set contains the pairs of endvertices of $\mathcal{P}$.

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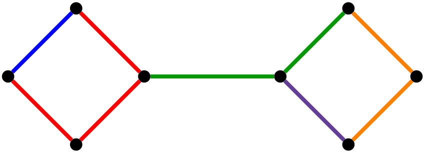

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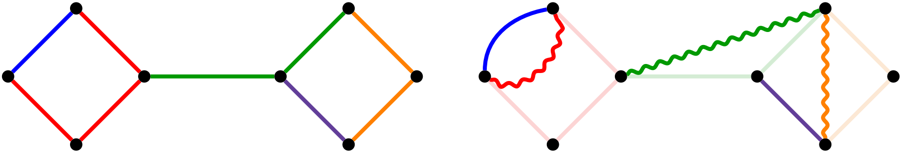

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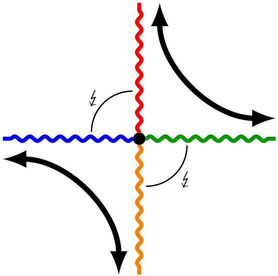
- For every $v \in V$, $d_H(v) = d_{H^*}(v)$.
- H **eulerian** $\Leftrightarrow$ H connected $+$ $d_H(v)$ even for every $v \in V$.
- H **q-path-graph** $\Leftrightarrow$ all paths of $\mathcal{P}$ have length q.
- H ($\leq$ **q**)**-path-graph** $\Leftrightarrow$ all paths of $\mathcal{P}$ have length at most q.
- H ($\geq$ **q**)**-path-graph** $\Leftrightarrow$ all paths of $\mathcal{P}$ have length at least q.
- H $(\mathbf{q_1}, \mathbf{q_2})$**-path-graph** $\Leftrightarrow$ all paths of $\mathcal{P}$ have length q_1 or q_2.

Previous example: disconnected ($\leq$ 3)-path-graph.

Conflicting paths

- **Conflicting paths** ⇔ paths sharing more than just one end.
- **Conflictless trail** ⇔ trail with no subsequent conflicting paths.
- *H* **conflictless eulerian** ⇔ *H* has a conflictless eulerian closed trail.



More terminology for conflicts

Definition: *multiplicity*

For distinct $w, v \in V$, the *multiplicity* of w *around* v is

$$\mathrm{mult}_v(w) := |\{P \in \mathcal{P}_v : w \in P\}|/|\mathcal{P}_v|.$$

The *multiplicity* of H is the maximum multiplicity of its vertices.

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Definitions: *conflict graph*, *conflict ratio*

For every $v \in V$, the *conflict graph* H_v is the graph on vertex set $\mathcal{P}_v$ in which P_1P_2 is an edge if P_1 and P_2 intersect. The *conflict ratio* of H is defined as

$$\max\{(\Delta(H_v) + 1)/|\mathcal{P}_v| : v \in V\}.$$

Remark: Every path is self-conflicting.

Eulerian closed trails and conflict ratio

Eulerian path-graph + reasonable conflict ratio $\Rightarrow$ conflictless eulerian closed trail.

Theorem

Every eulerian path-graph H with conflict ratio at most $1/8$ has a conflictless eulerian closed trail.

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Proof: Since the antidegree of every vertex in H_v is greater than $|\mathcal{P}_v|/2$, necessarily H_v admits a hamiltonian anticycle (by Dirac's Theorem). So there is a pairing $M_v = P_1P_2, P_3P_4, \ldots$ of the paths in $\mathcal{P}_v$ such that each pair is non-conflicting.

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Having such a pairing M_v for every $v \in V$ defines a set of conflictless closed trails $T_1, \ldots, T_t$, where a pair $\{P_i, P_{i+1}\}$ means that when entering at a vertex via P_i, we must leave via P_{i+1} (and vice-versa). If $t = 1$, we are done. Otherwise, we merge two trails so that t decreases.

Merging two conflictless closed trails

By our terminology, there is a $v \in V$ whose some paths of $\mathcal{P}_v$ belong to, say, T_1. We may assume that no more than half of its paths appear in T_1.

Assume $\{P_1, P_2\} \in M_v$ and $P_1, P_2 \in T_1$. Because P_1 and P_2 have degree less than $|\mathcal{P}_v|/8$ in H_v, there is a pair $\{P_3, P_4\}$ not in T_1 and such that P_1, P_2, P_3, P_4 are non-conflicting. Then replace $\{P_1, P_2\}$ and $\{P_3, P_4\}$ in M_v by $\{P_1, P_4\}$ and $\{P_2, P_3\}$. Then t decreases by 1, as claimed. ■

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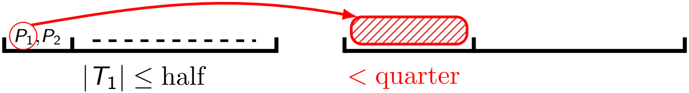

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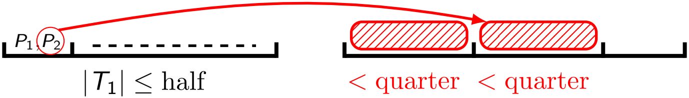

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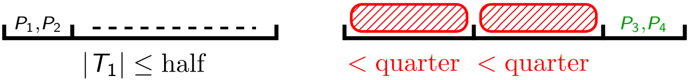

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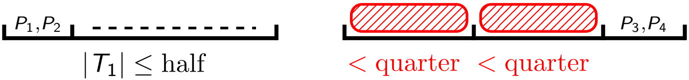

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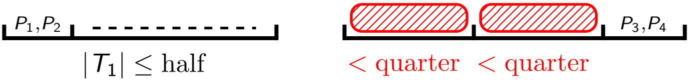
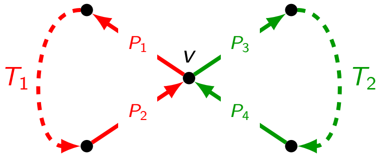
Assume $\{P_1, P_2\} \in M_v$ and $P_1, P_2 \in T_1$. Because P_1 and P_2 have degree less than $|\mathcal{P}_v|/8$ in H_v, there is a pair $\{P_3, P_4\}$ not in T_1 and such that P_1, P_2, P_3, P_4 are non-conflicting. Then replace $\{P_1, P_2\}$ and $\{P_3, P_4\}$ in M_v by $\{P_1, P_4\}$ and $\{P_2, P_3\}$. Then t decreases by 1, as claimed. ∎





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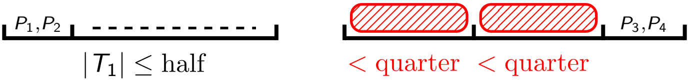

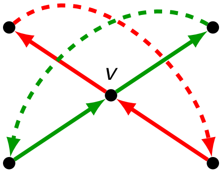

To larger paths with reasonable conflicts

Growing paths with still ‘reasonable’ path conflicts?

Lemma

Let q be some fixed positive integer and $c > 0$ be some real number such that $cq < 1/100$. Let $H = (V, \mathcal{P})$ be an α-dense q-path-graph of some graph G with multiplicity at most c and minimum degree k large with respect to $1/c$ and q. Then one can form an $\alpha/5$-dense $2q$-path-graph on G with multiplicity at most $16cq$.

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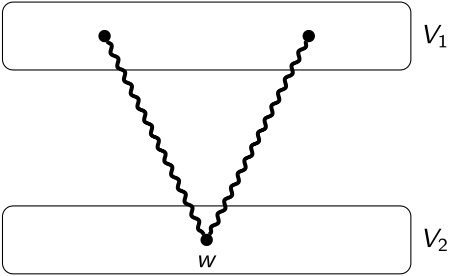
Proof idea: Consider a cut (V_1, V_2) of V maximizing the size of the set $\mathcal{P}'$ of edges of H^* 'between' V_1 and V_2. Let $H' = (V, \mathcal{P}')$. Then H' is $\alpha/2$-dense and has multiplicity at most $2c$. Now split H' into two $1/2$-fractions $H'_1 = (V, \mathcal{P}'_1)$ and $H'_2 = (V, \mathcal{P}'_2)$. These two path-graphs are $\alpha/4$-dense and have multiplicity at most $4c$. We use H'_1 only to form a $2q$-path graph on V_1 with required density (almost automatic) and multiplicity (much harder).

Overview of the proof

We iteratively want to randomly concatenate two q-paths meeting in V_2 at some vertex w to connect two vertices V_1 via a $2q$-paths.



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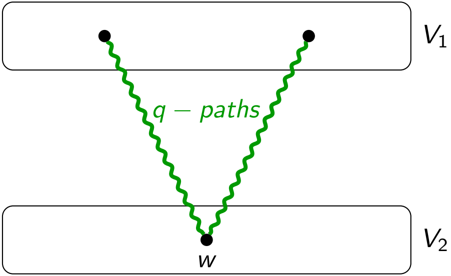

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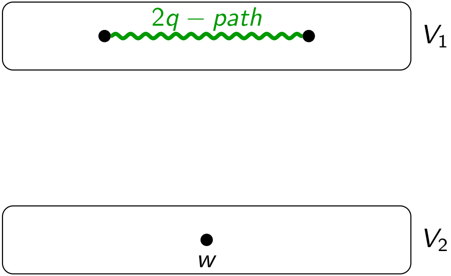

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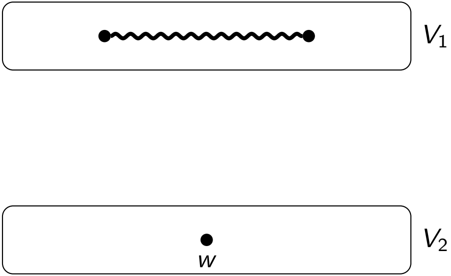

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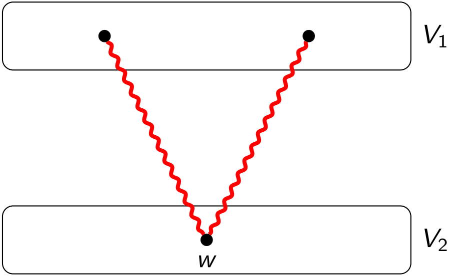

Problem: The two paths may be conflicting, and w can have degree so large that it carries too many path dependencies (making impossible *e.g.* the application of Lovász Local Lemma).

Pairing non-conflicting q-paths

Solution: Group the q-paths arriving at w into small subsets of non-conflicting paths $\rightarrow$ Possible because of the multiplicity assumption.

Generalization of Hajnal-Szemerédi Theorem

Let G be some graph of order n. Then, for every integer $t \geq \Delta(G) + 1$, the set $V(G)$ can be partitioned into $V_1, ..., V_t$ such that each V_i is an independent set of size $\lfloor \frac{n}{t} \rfloor$ or $\lceil \frac{n}{t} \rceil$.

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Now randomly pairing the q-paths arriving at every vertex w of V_2, we get a $2q$-path graph H_1'' spanning V_1. Multiplicity around every vertex v of V_1 is shown to be smaller than $16cq$ by combining LLL and Chernoff's bound.

Using H_2' instead H_1', we also obtain a $2q$-path graph H_2'' spanning V_2. ■

Constructing larger paths with reasonable conflicts

From repeated applications, we get:

Theorem

Let p be some integer and $0 < c < 1$ be some real number. There is an integer k depending on p and c such that every graph G with minimum degree at least k admits a $1/5^p$-dense 2^p-path graph H with multiplicity at most c.

Part 1: Introduction to the problem

Part 2: Fractions of graphs

Part 3: Path-graphs

Part 4: Constructing path-trees

Part 5: Using everything together

Part 6: Conclusion

Constructing $(1, \ell)$-trees

2-edge-connectivity + large degree $\Rightarrow$ $(1, \ell)$-(path-)tree with 'small' degree.

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Every 2-edge-connected graph G admits a subcubic spanning $(1, 2)$-tree.

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To every vertex v of G, we initially associate the empty $(1, 2)$-tree X on $\{v\}$. The procedure mainly consists in iteratively considering vertices at the highest depth, and 'merging' their corresponding $(1, 2)$-trees somehow. This is done with preserving 2-edge-connectivity.

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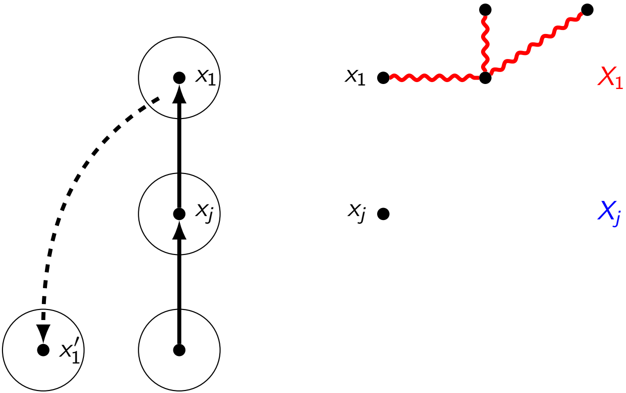
The key to respect the degree condition is that there are only two possibilities for increasing the degree of a vertex in a merged $(1, 2)$-tree, namely by adding its incident forward and backward edges. ■

Sample cases – One child



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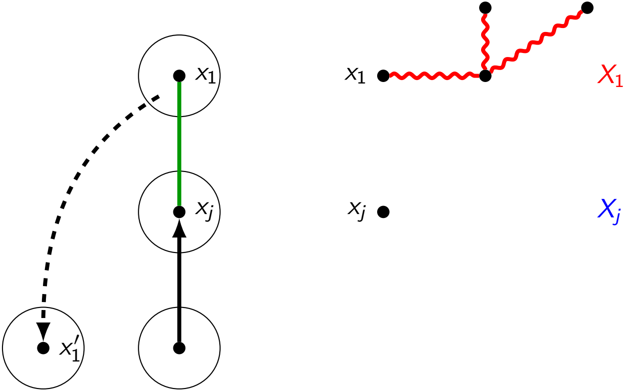

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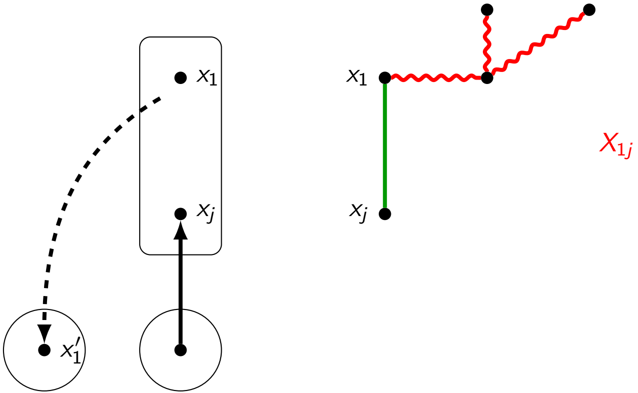

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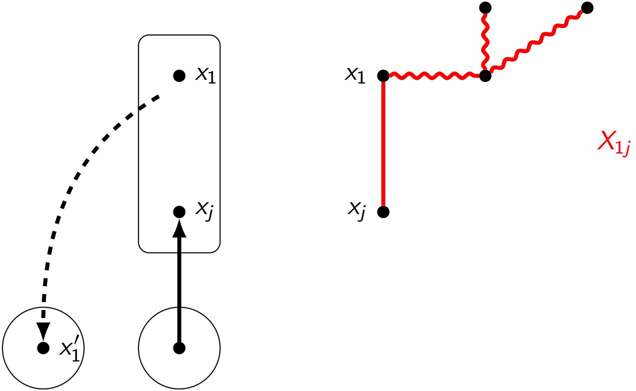
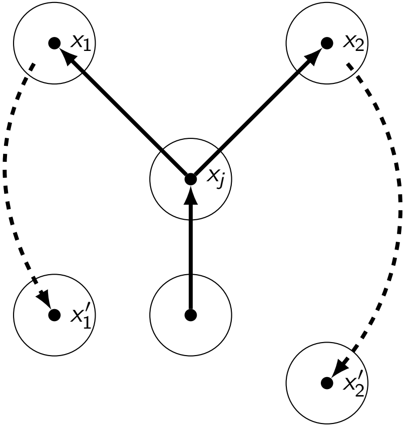
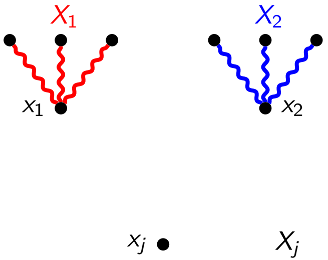


Sample cases – Two children





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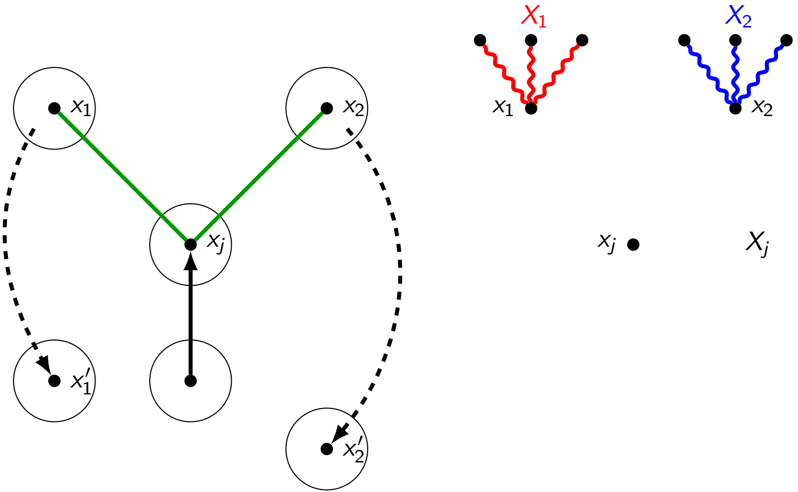
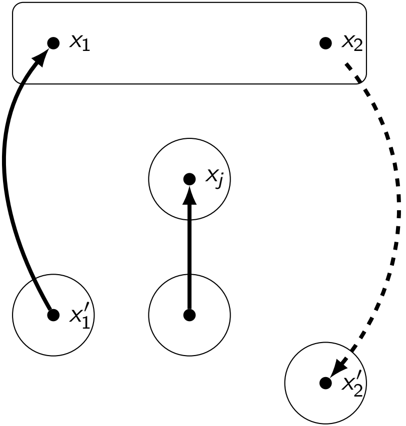
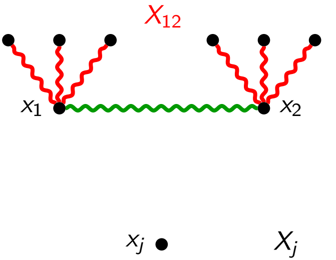



x_1
x_2
x_j
x'_1
x'_2


X_{12}
x_1
x_2
x_j
X_j

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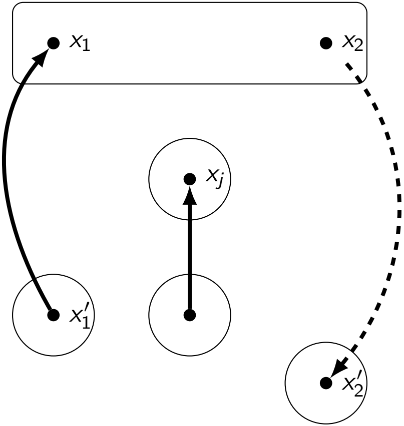

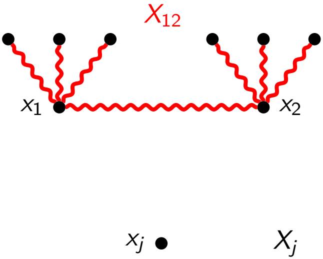

From $(1,2)$-trees to $(1,k)$-trees

spanning $(1,k)$-tree + disjoint 'source' of degree = spanning $(1,k+1)$-tree.

Theorem

Let T be a spanning $(1,k)$-tree of some graph $G=(V,E)$, and let H be some additional graph on V, edge-disjoint from G, and satisfying $d_H(v) \geq 2(d_T(v)+2k)$ for every $v \in V$. Then $G \cup H$ is spanned by a $(1,k+1)$-tree T'.

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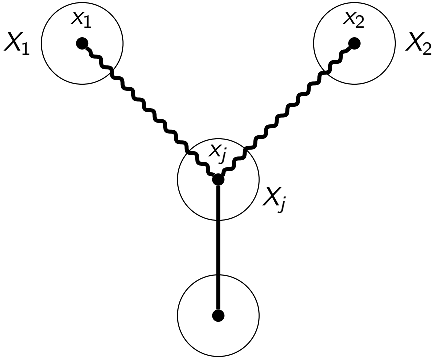
To make sure that the edges of H are equitably used and not 'saturated' by some vertex, we orient them in a balanced way beforehand (hence defining *private edges* for every vertex). ■

Sample case – Two children



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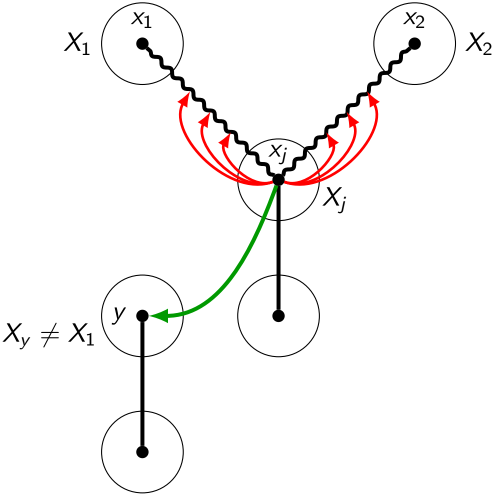

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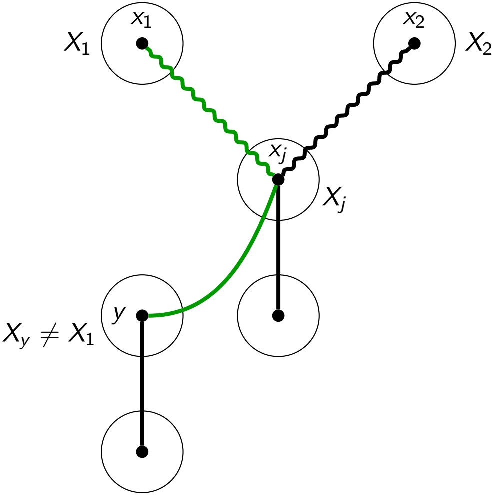

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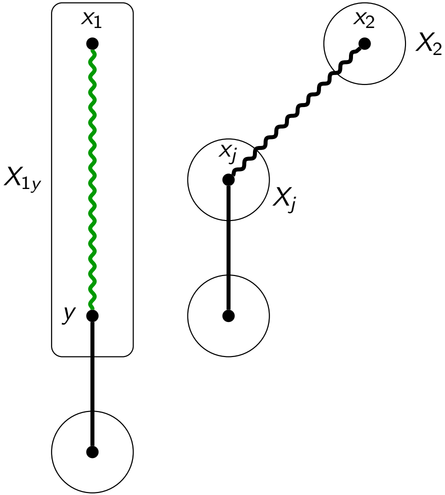

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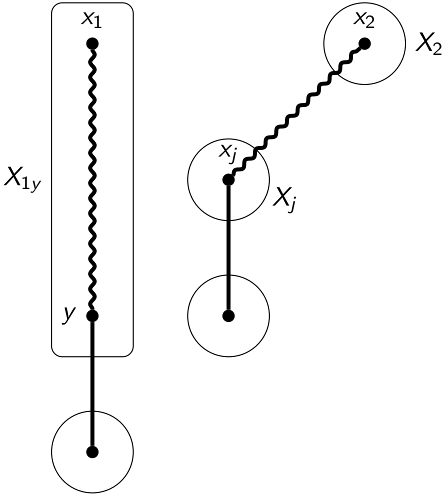

$(1, k)$-trees – Summary

2-edge-connectivity + disjoint 'source' of degree = spanning $(1, \ell)$-tree.

Corollary

For every $\ell \geq 1$, there exists k_ℓ such that if $G = (V, E)$ is a 2-edge-connected graph and H is some additional graph on V with minimum degree k_ℓ, then $G \cup H$ is spanned by a $(1, \ell + 1)$-tree T where $d_T(v) \leq d_H(v)$ for every $v \in V$.

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Proof: First deduce a subcubic $(1, 2)$-tree T_2 spanning G. Then consider a sequence of disjoint small fraction $H_1, ..., H_{\ell-1}$ of H, where each H_i is an ε_i-fraction of H. By the assumption on k_ℓ, we can assume $\varepsilon_{i+1} \geq 4\varepsilon_i$.

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Using H_1, from T_2 we can deduce a $(1, 3)$-tree T_3 spanning G. Note that $d_{T_3}(v) \leq d_{T_2}(v) + d_{H_1}(v)$ for every vertex $v \in V$. Since $4d_{H_1}(v) \leq d_{H_2}(v)$, we can use H_2 to extend T_3 to a $(1, 4)$-tree T_4 spanning G. Due to the choice of the ε_i's, this process can be repeated until we get T. ■

Path-trees in bipartite graphs

Path-trees with paths of lengths multiple of ℓ?

Theorem

For every even $\ell \geq 2$, there exists k_ℓ such that if $G = (V, E)$ is a 2-edge-connected bipartite graph with vertex partition (A, B) and H is some additional bipartite graph with vertex bipartition (A, B) and minimum degree k_ℓ, then $G \cup H$ admits a $(\ell, 2\ell)$-tree T spanning A where $d_T(v) \leq d_H(v)$ for every $v \in V$.

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We may assume $4\varepsilon \leq 1/5^p$, where $p = \lceil \log_2 \ell \rceil$. Also, we can deduce a $1/5^p$-dense 2^p-path-graph H' on H with multiplicity at most $1/16.2^p$. Orienting H'^* in a balanced way, every vertex v is the origin of at least $d_H(v)/2.5^p$ private 2^p-paths of H' (with multiplicity at most $1/8.2^p$). Furthermore, $2^p \geq \ell - 1$.

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To obtain T, we once again start from every vertex v of A being its own $(\ell, 2\ell)$-tree X_v. We then iteratively consider a leaf u in T' with parent x, and concatenate the path associated to ux with some private $(\ell - 1)$-path of x. This forms an ℓ or 2ℓ-path. ■

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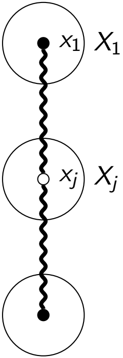

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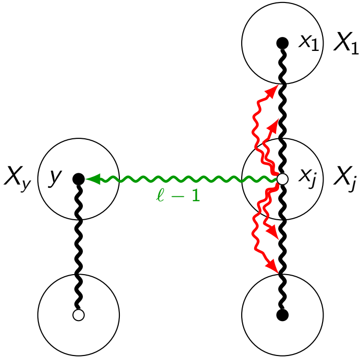

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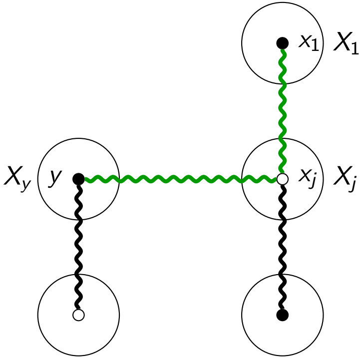

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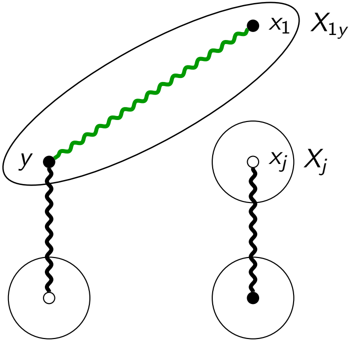

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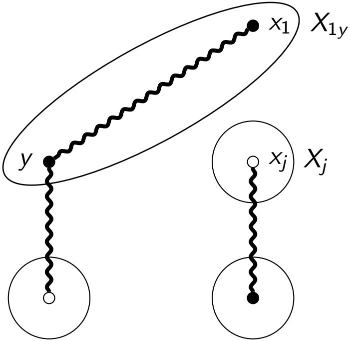

Overview, second try

Main ideas:

1. See G as a 'convenient' system H of induced paths.
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Refined to...

1. Turn G into a $(\geq \ell)$-path graph H.

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Remark: We may suppose ℓ is even.

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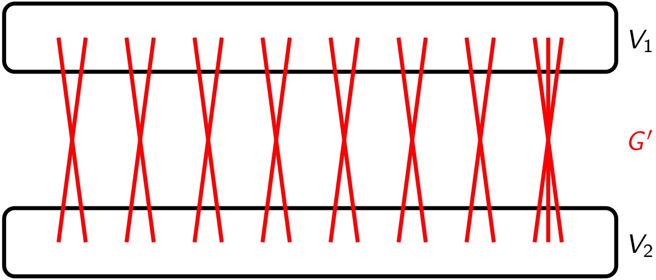

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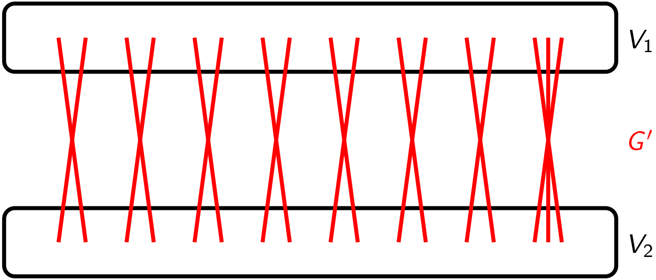

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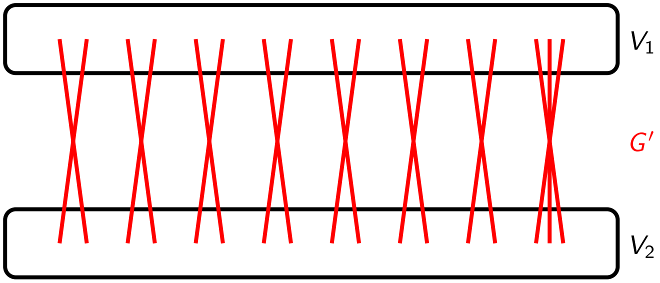

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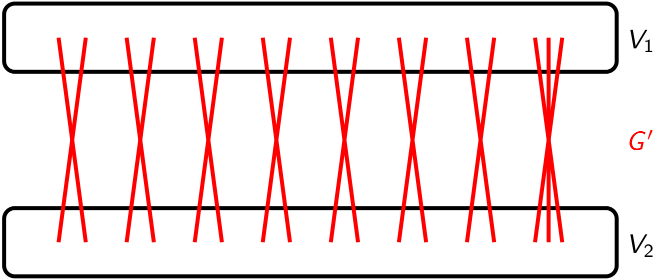

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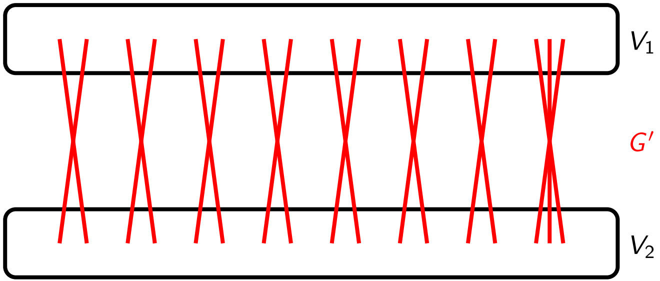

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Set $H_1 := T_1 \cup T_2$, $H_2 := T_3 \cup T_4$, $H_3 := T_5 \cup T_6$, $H_4 := T_7 \cup T_8$, and H_5 the rest.

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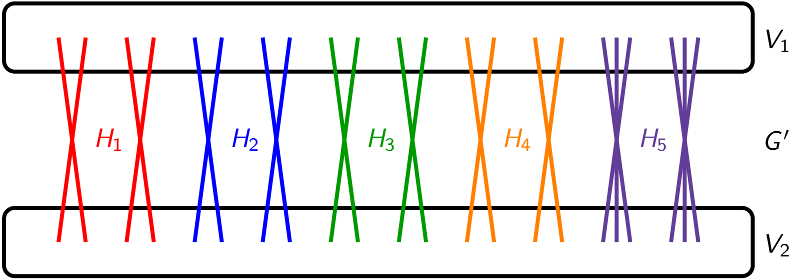

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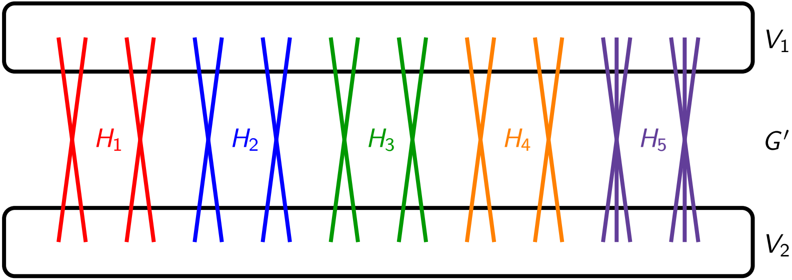

$$G' = \underbrace{H_1, H_2, H_3, H_4}_{} + \underbrace{H_5}_{}$$

- connected
- bridgeless
- 1/2-sparse

- 1/2-dense

Creating $(\ell, 2\ell)$-trees over V_1 and V_2

H_1 + tiny c-fraction of H_5 = c-sparse $(\ell, 2\ell)$-tree T_1' spanning V_1.
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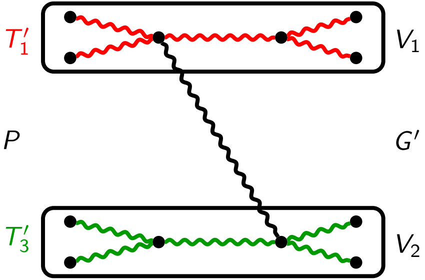
$T = T_1' \cup T_3' \cup P$ is a sparse $(\geq \ell)$-tree spanning G.



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We may suppose H_5 is still $1/2$-dense in G'. Because of the degree assumption, it has a $1/5^p$-dense 2^p-path-graph H with multiplicity at most c , where p satisfies $\ell \leq 2^p < 2\ell$.

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So far, G is decomposed, saved some ℓ-paths, into:

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- R: • rest

As long as possible, remove ℓ-paths from R. Call G_R what remains.

Theorem [Thomassen – 2008]

G_R admits a $(< \ell)$-path-graph with maximum degree at most $\ell - 1$.

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Consider a 'tiny' $4c\ell^3$-fraction of H' of H, and orient H'^* in a balanced way. Then every vertex v is the origin of a private set of $K = 2c\ell^3 d_H(v)$ paths $P'_1, ..., P'_K$ in H'.

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Call H'_R the resulting $(\geq \ell)$-path graph. In particular, this graph is sparse in H.

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Our objects are:

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Now, since the multiplicity of H is arbitrarily small and $T \cup T_2' \cup T_4' \cup H_R'$ is, say, $4c\ell^3$-sparse, H_F is a $(\leq 3\ell)$-path-graph with multiplicity less than $1/24\ell$. Then a conflictless eulerian closed trail exists in H_F. Going along it, we finish the decomposition into ℓ-paths. ■

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Thank you for your attention.