Introducing and Extracting Irregularity in Graphs

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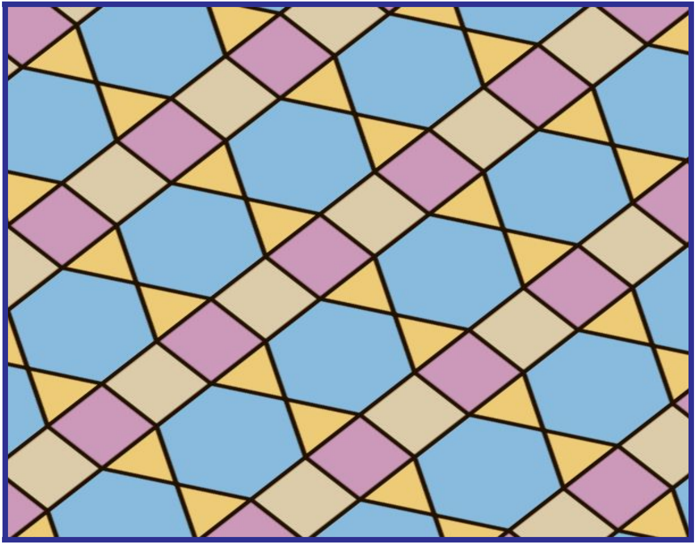
What is graph irregularity?

“Regular” versus “Chaotic” patterns



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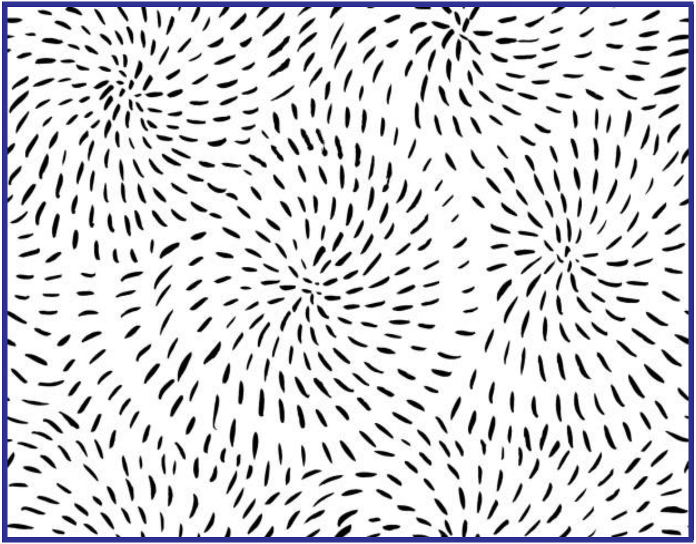

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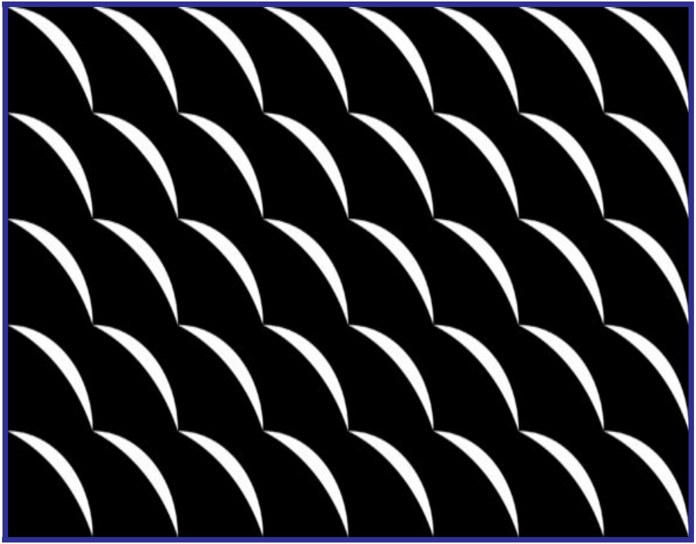

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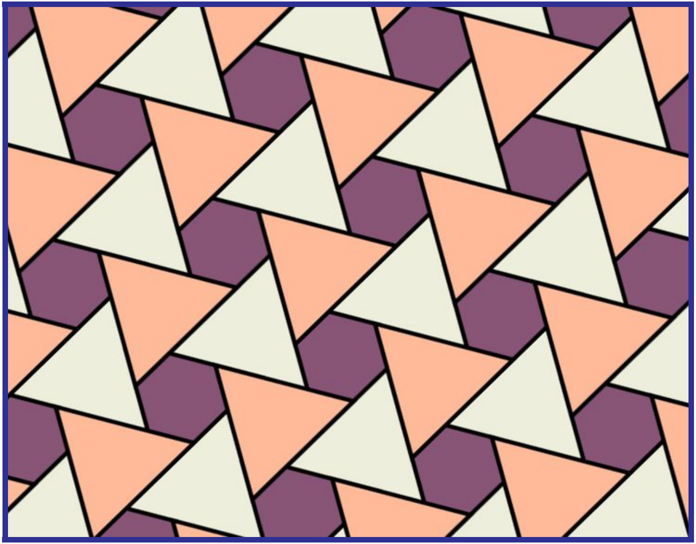

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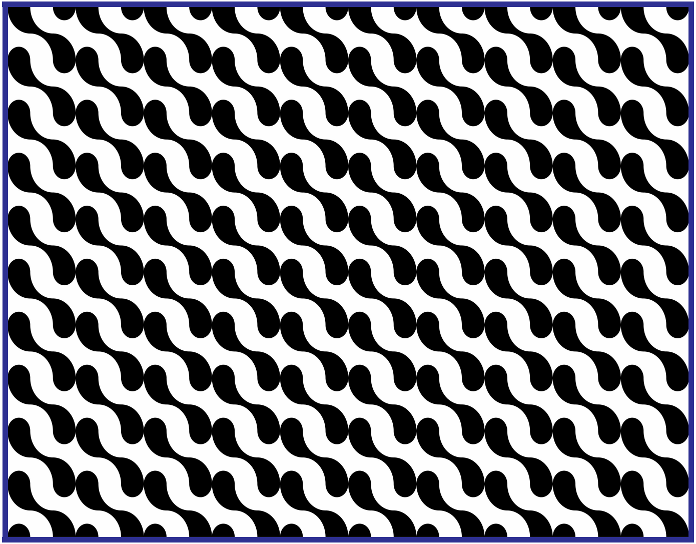

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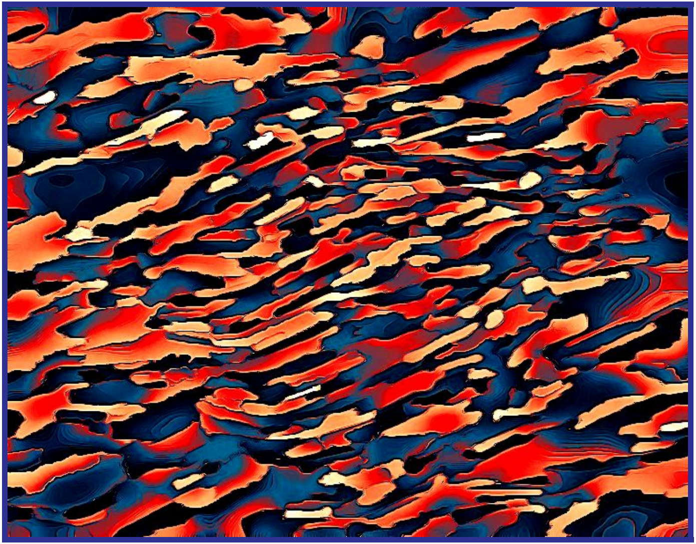

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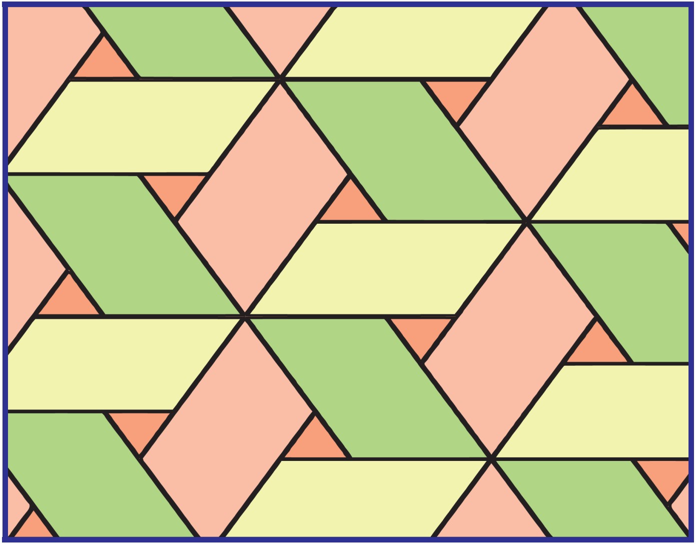

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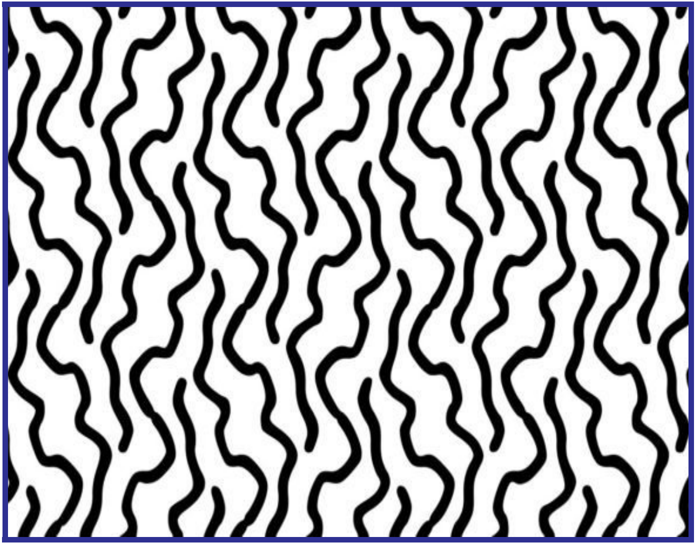

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Blend of **evolutionary**, **cognitive**, **cultural**, and **psychological factors** influencing our **perceptions and preferences**



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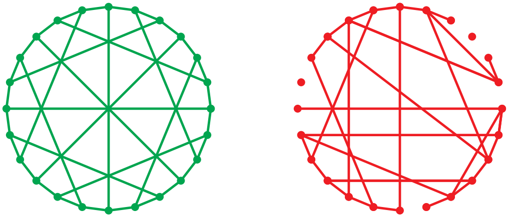
- **Several notions** of symmetry, regularity, order:
  - **regularity w.r.t. degrees**
  - vertex/edge transitivity, graph automorphisms, etc.
  - results such as Szemerédi’s Regularity Lemma
  - existence of “nice” plane embeddings
  - etc.



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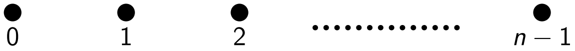

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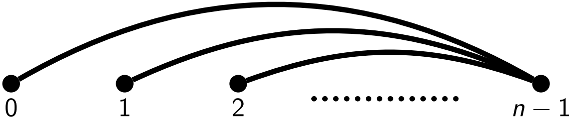

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   - Related **decompositional** approaches and problems

Warning: Very wide field, will not be exhaustive (we can discuss later ☺)

Reaching irregularity through edge multiplications

Making simple graphs totally irregular (1986)

IRREGULAR NETWORKS

Gary Chartrand[1], Western Michigan University
Michael S. Jacobson, University of Louisville
Jenö Lehel, Computer and Automation Institute, Hungarian Academy of Sciences, Budapest
Ortrud R. Oellermann, Western Michigan University
Sergio Ruiz, Universidad Católica de Valparaíso, Chile
Farrokh Saba, Western Michigan University

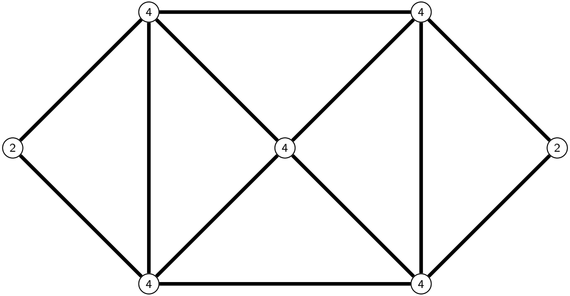
ABSTRACT

A network N is a graph in which each edge is assigned a positive integer weight. The degree of a vertex in N is the sum of the weights of its incident edges. A network is irregular if its vertices have distinct degrees. The strength of a network N is the maximum weight among the edges of N. The irregularity strength s(G) of a graph G is the minimum strength among the irregular networks having G as an underlying graph. It is shown that s(G) is defined for every connected graph G of order $p \geq 3$ and that $s(G) \leq 2p - 3$. Further, if N is a network of strength at least 2, then there exists an irregular network having the same strength as N and containing N as an induced subnetwork.

1. Introduction

A graph G is <u>regular</u> if its vertices have the same degree; G is <u>irregular</u> if its vertices have distinct degrees. While the literature abounds with results about regular graphs, it is well known that nontrivial irregular graphs fail even to exist. Such is not the case for multigraphs, however. For example, the multigraph of Figure 1(a) is irregular, having vertices of degrees 3, 4 and 5.

Sample example



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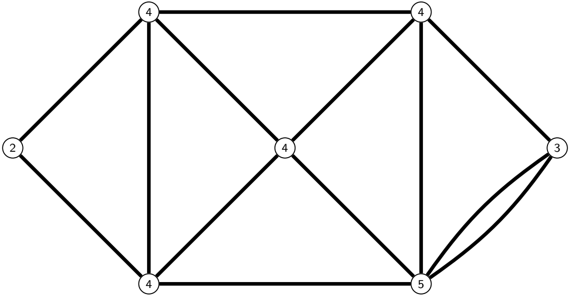

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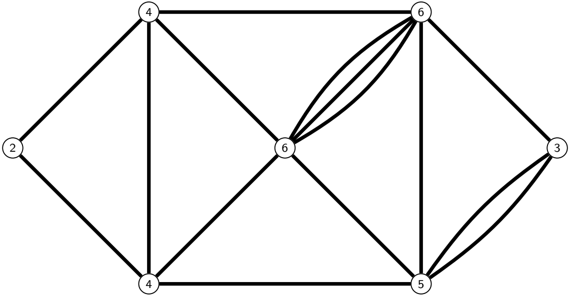

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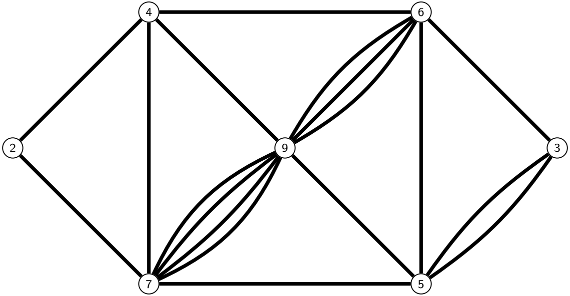

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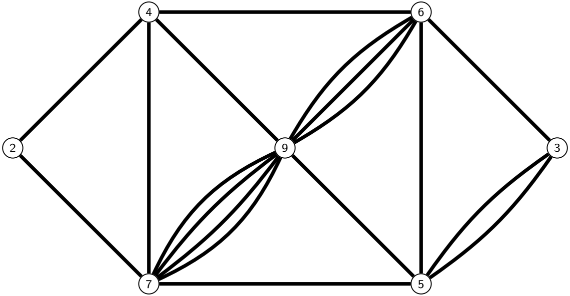

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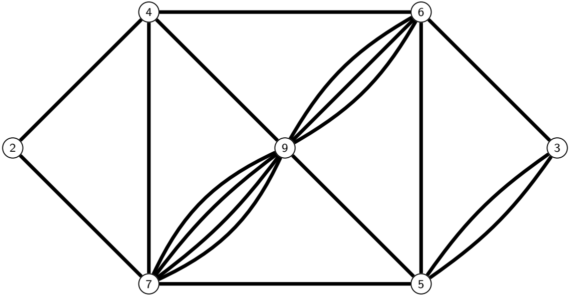

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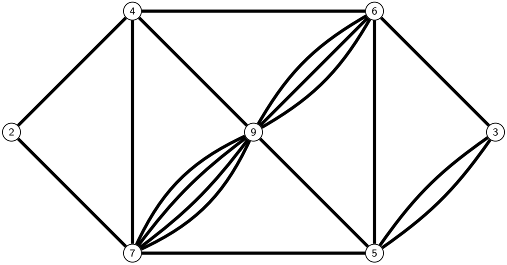
- Preserves the **original structure**
- Chartrand *et al.*: avoid "exploding" an edge too much?
- **Above:** every edge $\rightarrow \leq 4$ parallel edges; what about ≤ 3?

Another take on the problem

Remark: Previous problem a bit tedious to study



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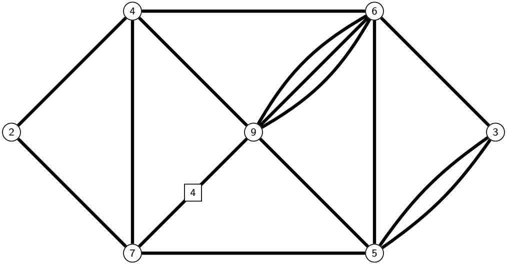

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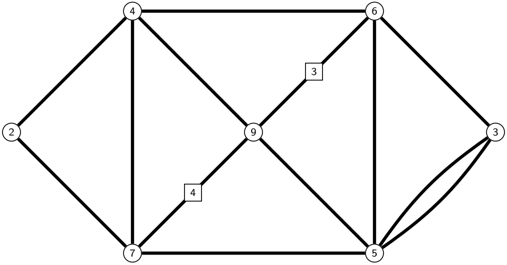

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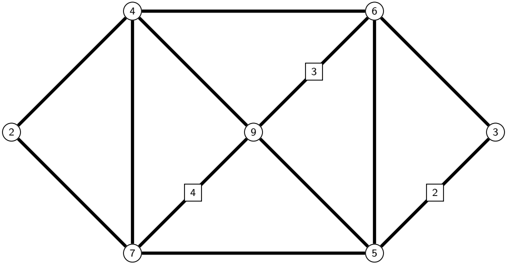

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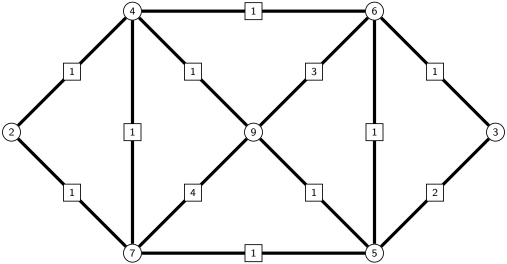

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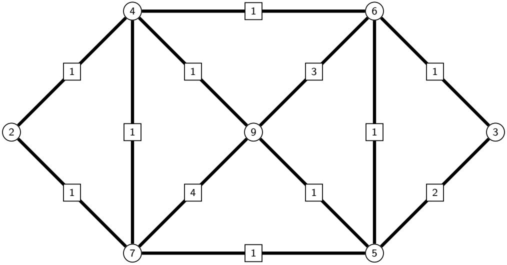

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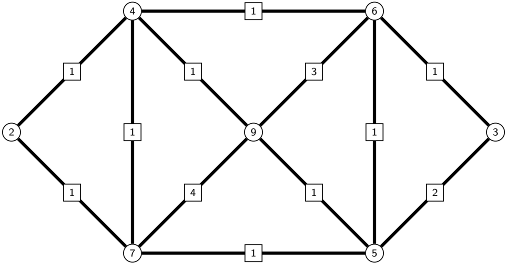

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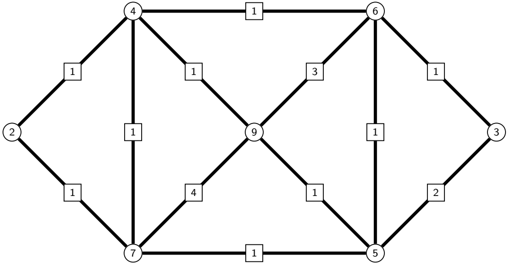

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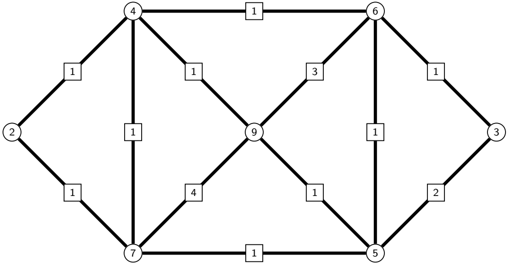

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- Degrees → Incident sums
- T.i. multigraph → **T.i. labelling**
- Minimising max. edge "explosion" → Minimising max. label
- **Irregularity strength s(G)** of G: this minimum

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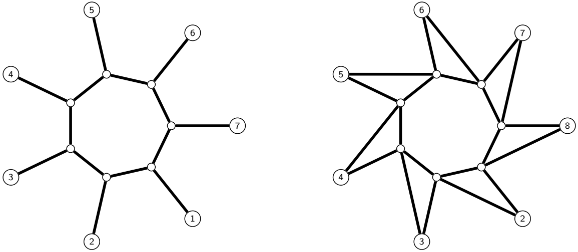
- **s(G) not bounded by an absolute constant** $k \geq 1$

For any $x \geq 0$, set **nb(x)** as the # of degree-x vertices; then, need:

- $\mathrm{nb}(1) \leq k$ for $x = 1$, sums in $\{1, \dots, k\}$
- $\mathrm{nb}(2) \leq 2k - 1$ for $x = 2$, sums in $\{2, \dots, 2k\}$
- $\mathrm{nb}(3) \leq 3k - 2$ for $x = 3$, sums in $\{3, \dots, 3k\}$
- etc.



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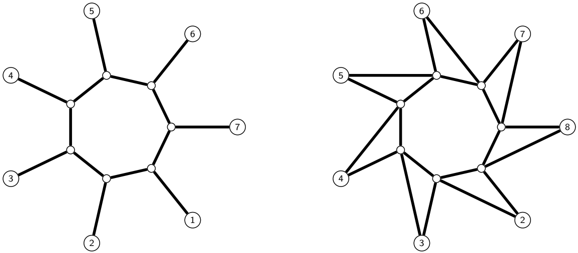

But vertices with different, yet close degrees can also "collide" ☹

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- Variants (local, total, etc.)

Considering other types of graph irregularity

Highly irregular graphs (1987)

Highly Irregular Graphs*

Yousef Alavi
Gary Chartrand
WESTERN MICHIGAN UNIVERSITY

F. R. K. Chung
BELL COMMUNICATIONS RESEARCH

Paul Erdös
HUNGARIAN ACADEMY OF SCIENCES

R. L. Graham
AT&T BELL LABORATORIES

Ortrud R. Oellermann
WESTERN MICHIGAN UNIVERSITY

ABSTRACT

A connected graph is highly irregular if each of its vertices is adjacent only to vertices with distinct degrees. In this paper we investigate several problems concerning the existence and enumeration of highly irregular graphs as well as their independence numbers, with particular focus on the corresponding problems for highly irregular trees.

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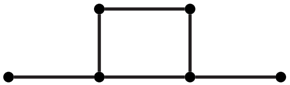
High irregularity (h.i.) = No vertex has two neighbours with the same degree



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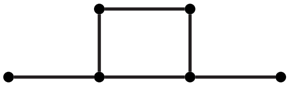

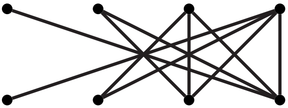

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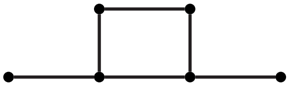

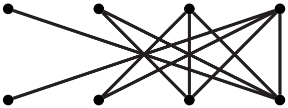

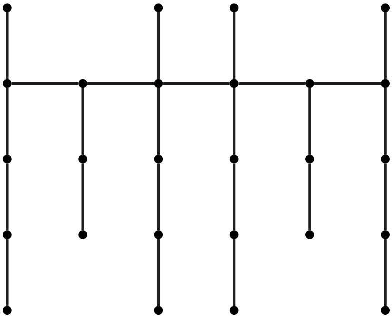

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- **Proportionality for fixed order**
- Independence number
- Degree sequence and properties

Highly irregular graphs

Studied aspects (mostly [Alavi, *et al.*, 1987]):

- **Existence for all n $\notin \{3, 5, 7\}$**
- **Maximum size w.r.t. order**
- Minimum order w.r.t. containing a prescribed graph
- **Minimum order for trees with given maximum degree**
- **Proportionality for fixed order**
- Independence number
- Degree sequence and properties
- Etc.

Locally irregular graphs (2015)

On decomposing regular graphs into locally irregular subgraphs

O. Baudon[a,b], J. Bensmail[a,b], J. Przybyło[c,1], M. Woźniak[c]

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A R T I C L E   I N F O

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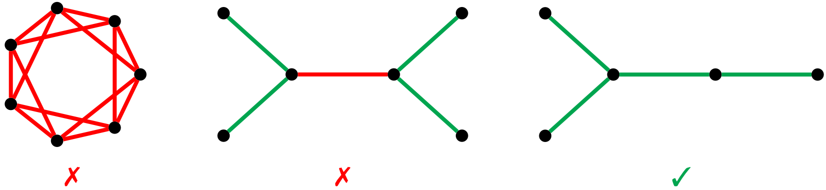
A B S T R A C T

A *locally irregular graph* is a graph whose adjacent vertices have distinct degrees. We say that a graph *G can be decomposed into k locally irregular subgraphs* if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph in G. We characterize all connected graphs which cannot be decomposed into locally irregular subgraphs. These are all of odd size and include paths, cycles and a special class of graphs of maximum degree 3. Moreover we conjecture that apart from these exceptions all other connected graphs can be decomposed into 3 locally irregular subgraphs. Using a combination of a probabilistic approach and some known theorems on degree constrained subgraphs of a given graph, we prove this statement to hold for all regular graphs of degree at least 10^7. We also support this conjecture by showing that decompositions into three or two such subgraphs might be indicated e.g. for some bipartite graphs (including trees), complete graphs and cartesian products of graphs with this property (hypercubes for instance). We also investigate a total version of this problem, where in some sense also the vertices are being prescribed to particular subgraphs of a decomposition. Both the concepts are closely related to the known 1-2-3 Conjecture and 1-2 Conjecture, respectively, and other similar problems concerning edge colourings. In particular, we improve the result of Addario-Berry et al. (2005) in the case of regular graphs.

Locally irregular graphs

Local irregularity (l.i.) = No two adjacent vertices with the same degree



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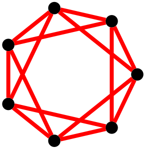

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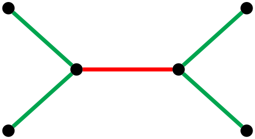

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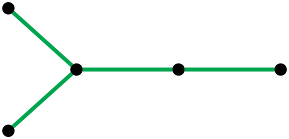

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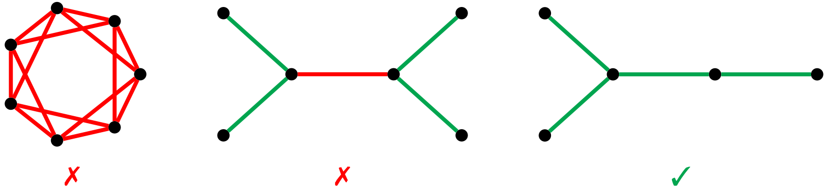

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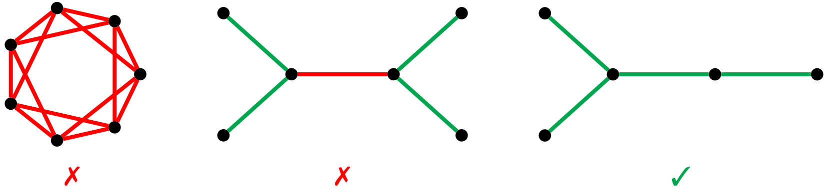

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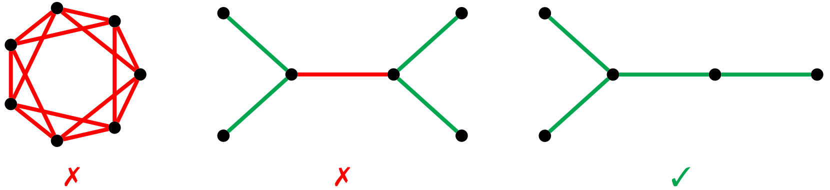

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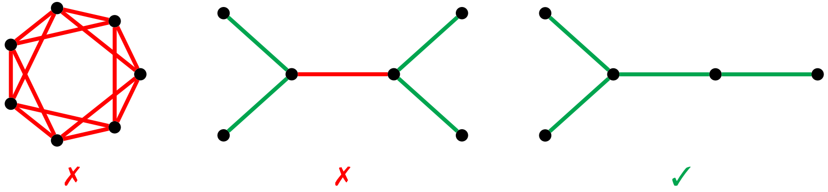

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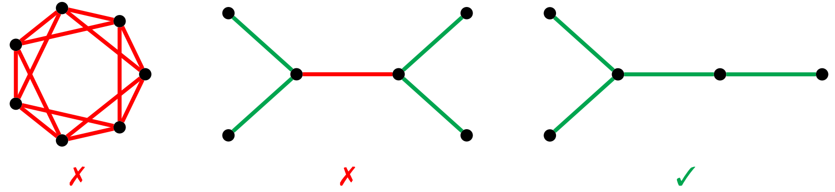

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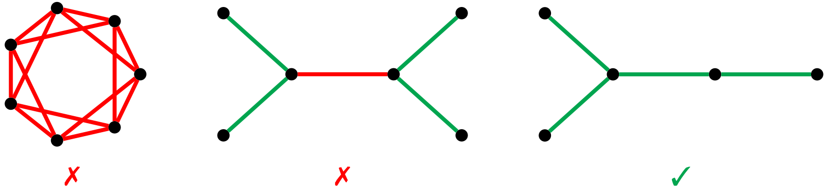

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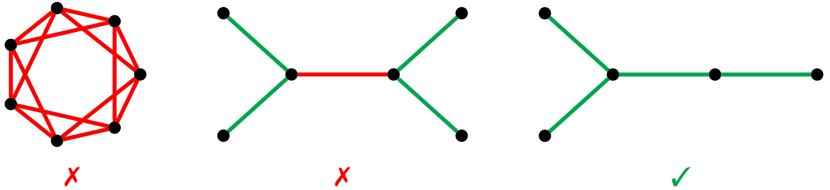

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- **T.i.** $\Rightarrow$ **h.i.** $+$ **l.i.** for multigraphs...
- Probably other notions of irregularity in literature

Distinguishing labellings

Generalising Chartrand et al's problem to high and local irregularity?

	Total irregularity	High irregularity	Local irregularity
Labellings	t.i. labellings **irregularity strength**	h.i. labellings	l.i. labellings **1-2-3 Conjecture**

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- **1-2-3 Conjecture** [Karoński, Łuczak, Thomason, 2004]
  - **Possible for all connected graphs but K_2 ☹**
  - **Conjecture:** labels $1, 2, 3$ suffice for all connected graphs $G \neq K_2$
  - **Deciding if $1, 2$ suffice is NP-complete [Dudek, Wajc, 2011]**
  - **Many derived variants and related questions**
  - Hold on for what else we know ☺ ...

Three variants of the 1-2-3 Conjecture

- **Original variant:**
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- **Note:** other variants exist as well (for sets, sequences, etc.)

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Product version $\sim$ Multiset version with a neutral label

How bounds progressed

[Karoński, Łuczak, Thomason]

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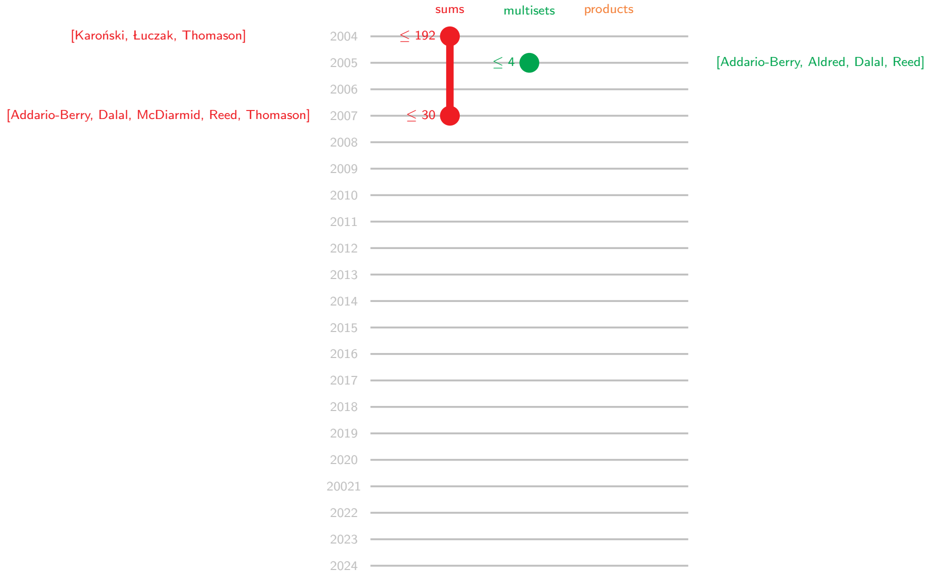

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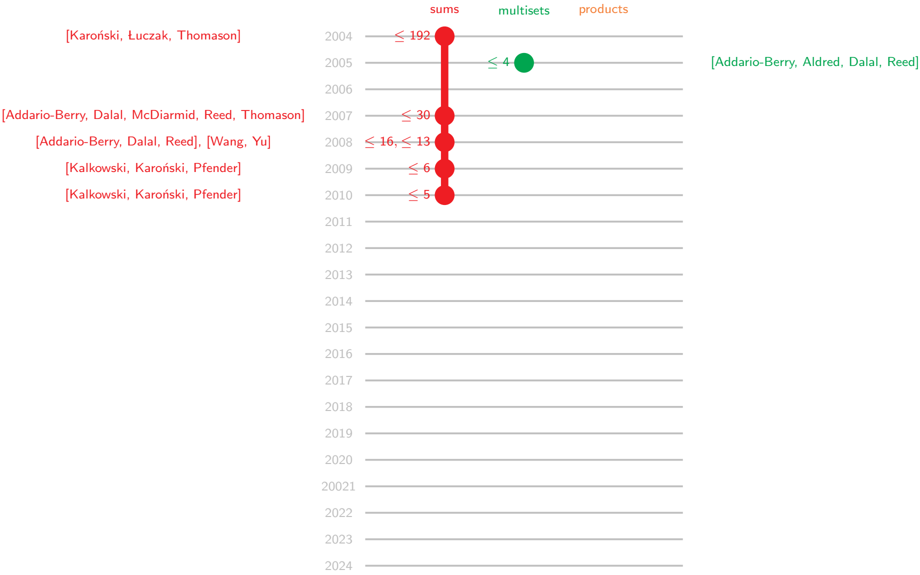

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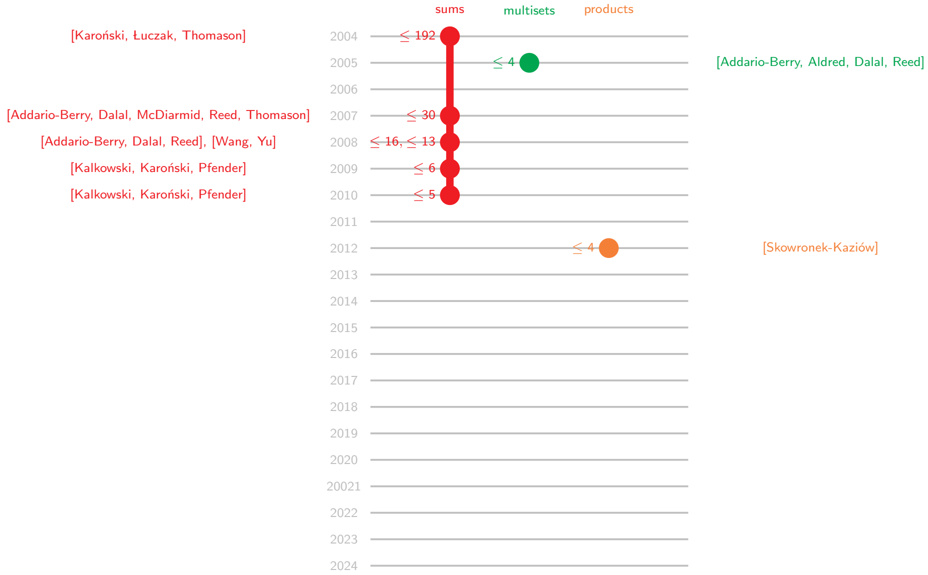

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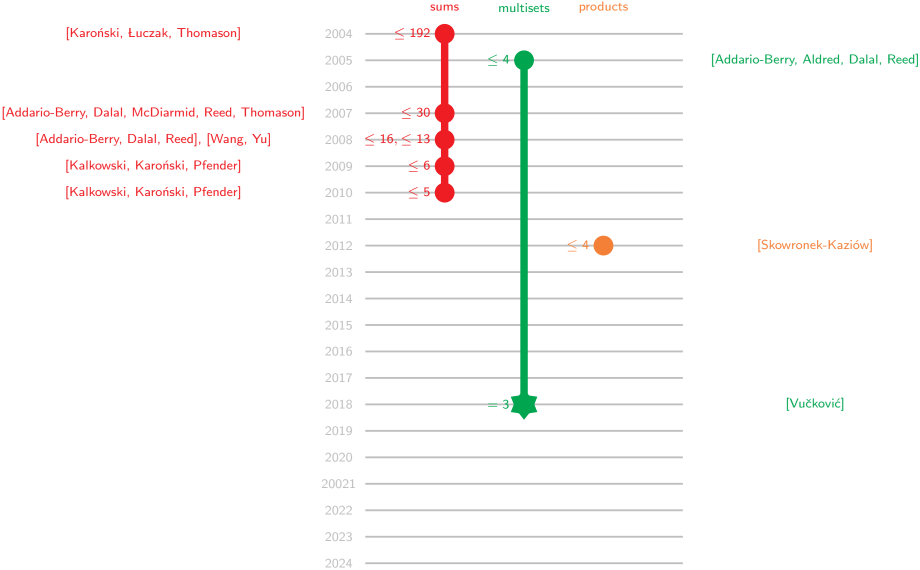

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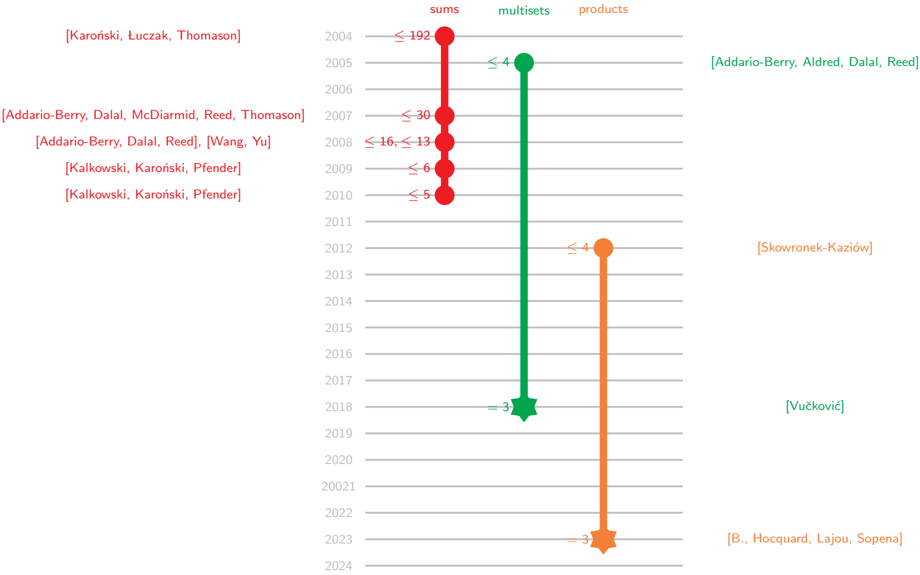

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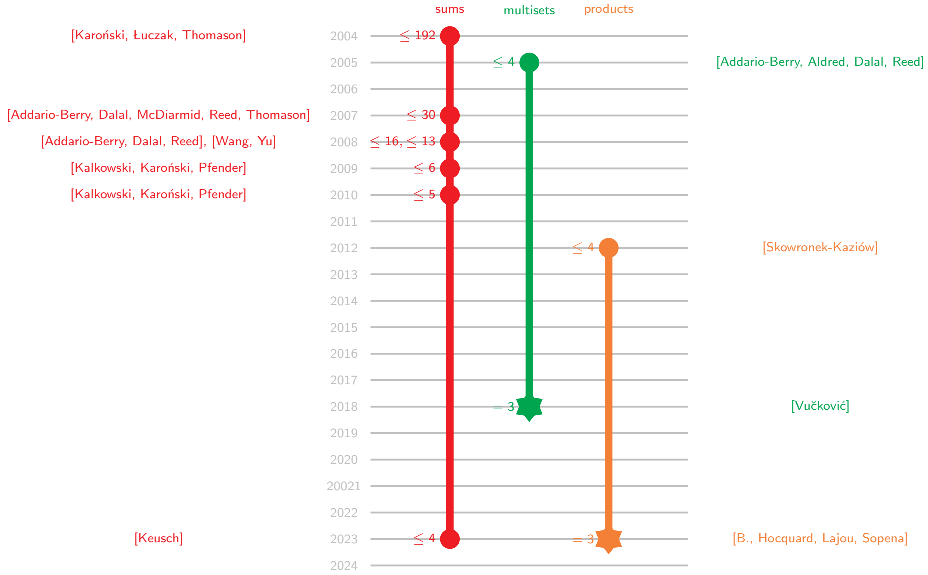

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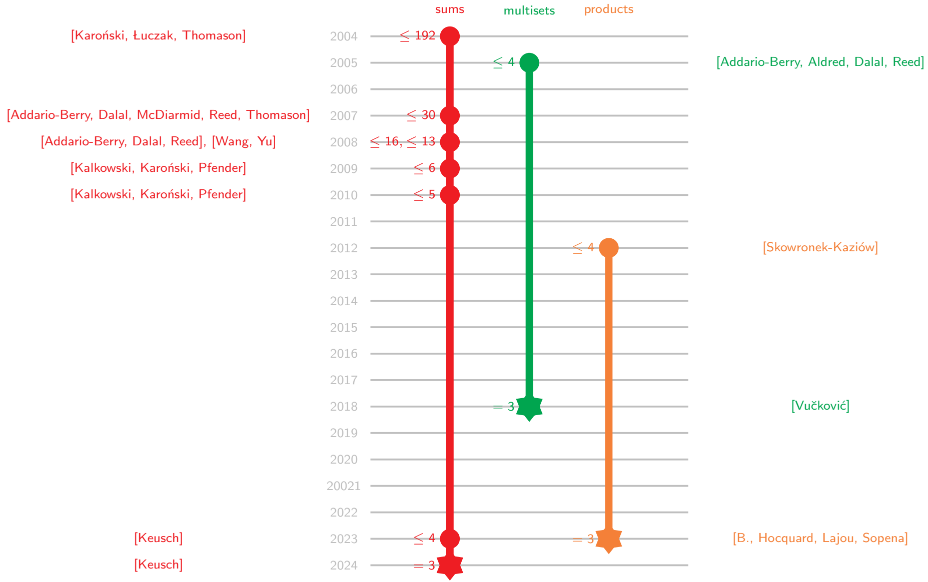

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In what follows: a flavour of the proof for products ☺

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For every connected graph different from K_2, we can assign labels $1, 2, 3$ to the edges so that neighbours are distinguished by their incident products.

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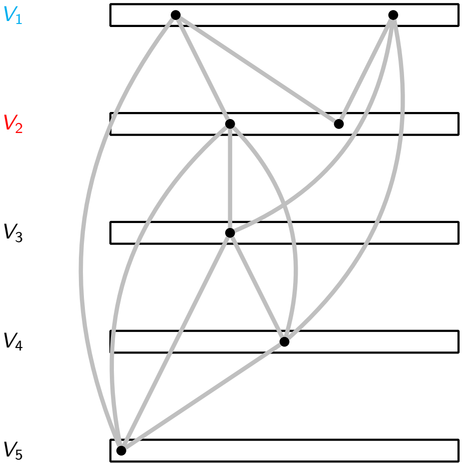
The type of labelling we want by the end of Step 2



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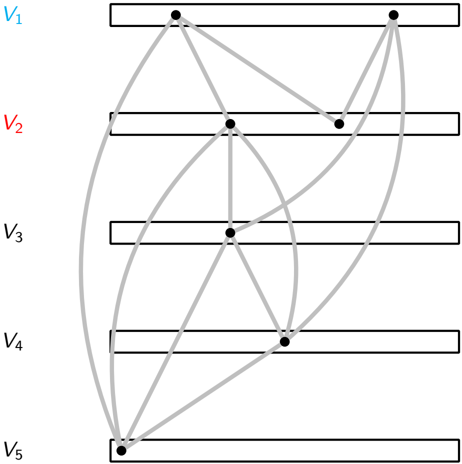

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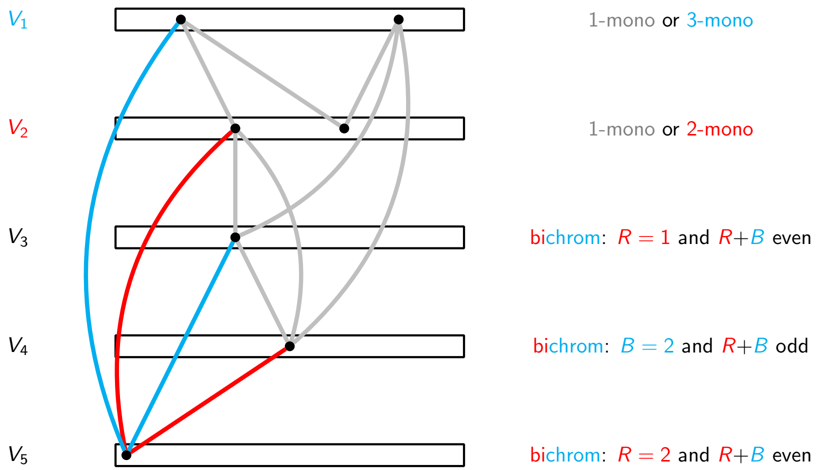

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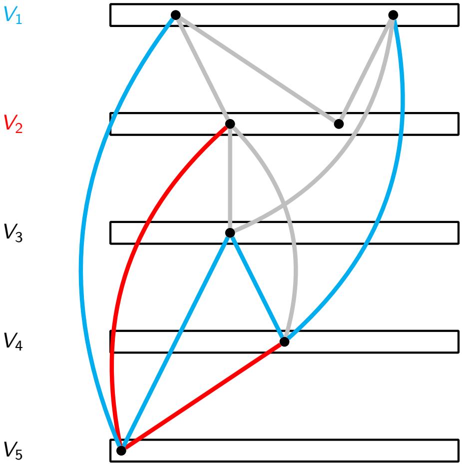

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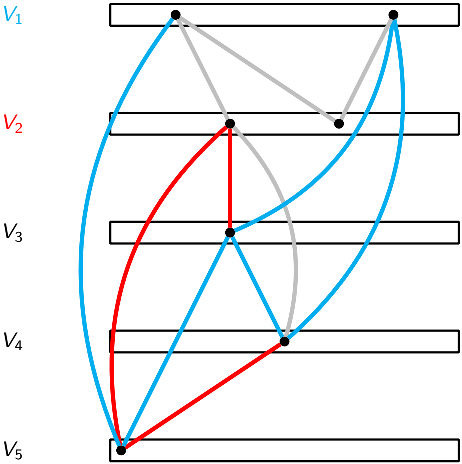

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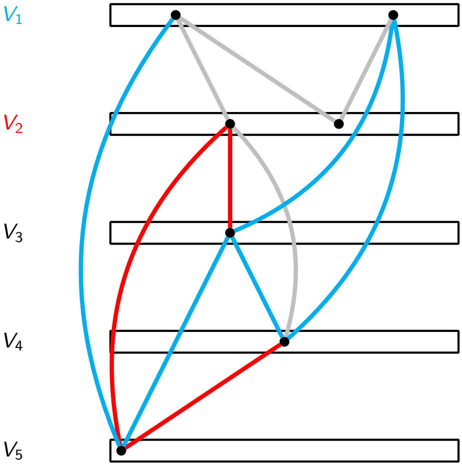

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Note:

- No conflict between odd layers; same for even layers
- Same between odd layers and even layers (except for 1-mono across (V_1, V_2))
- No special vertex ($B = 1$ and $R+B$ odd)

Decomposing graphs into irregular graphs

Where these concerns come from

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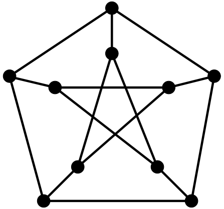

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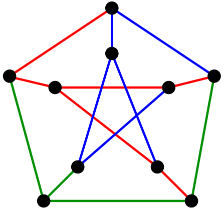

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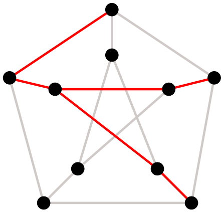

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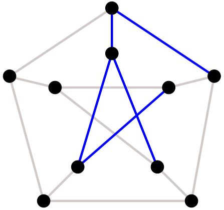

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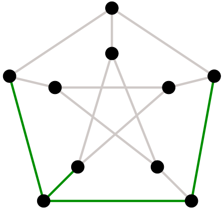

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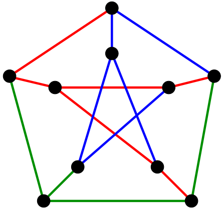

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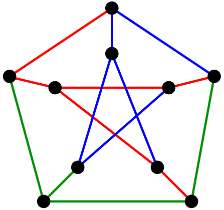

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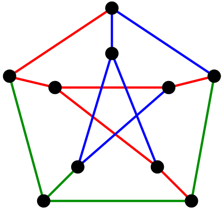

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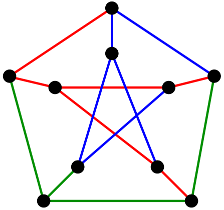

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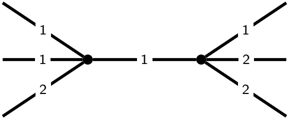

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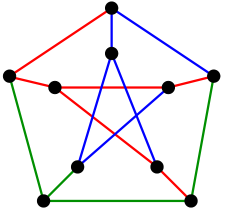

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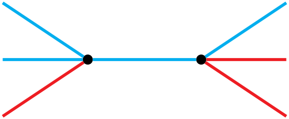

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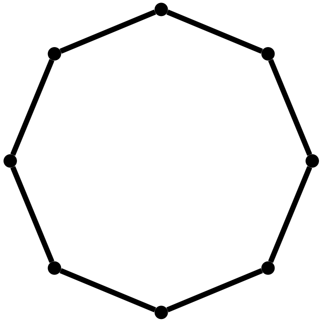

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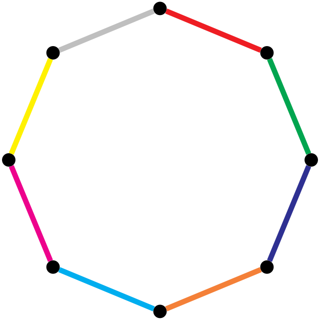

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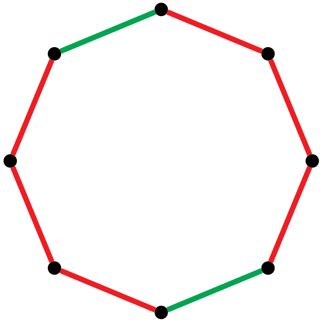

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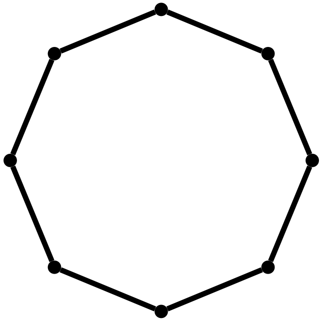

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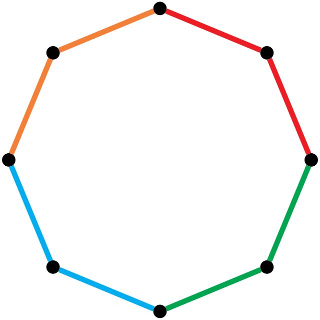

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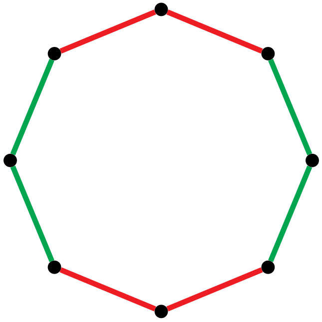

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	Total irregularity	**High irregularity**	**Local irregularity**
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- **Locally Irregular Decompositions** [Baudon, B., Przybyło, Woźniak, 2015]
  - **Three infinite classes of exceptions** ☹
  - **Conjecture:** three colours suffice for all graphs (but one + exceptions)
  - **Proved true for some classes of graphs**
  - **Best result to date:** 220 colours suffice!
  - **Deciding for two colours is NP-complete [Baudon, B., Sopena, 2015]**

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Theorem – B., Merker, Thomassen, 2016

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- Decomposing degenerate graphs and graphs with large minimum degree

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Side note, main point:

Theorem – Folklore

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When a connected graph has odd size:

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In every connected, decomposable graph of odd size, there is a  or a  whose deletion yields even-size connected components only.

Thus, "cost" of 1 for reducing to even-size graphs (and avoiding troubles ☺)

Second ingredient: Decomposing even-size bipartite graphs

Main result:

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- **Actually, "almost" sufficient, might be a (single) faulty vertex ☹**
  - If no cycle through it, all incident edges are bridges ⇒ Dedicated arguments
  - Otherwise, remove a cycle C through it + a degree-decreasing path P from it
  - ⇒ Structure on $E(C) \cup E(P)$ decomposes into at most 4 l.i. graphs

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  - Otherwise, remove a cycle C through it + a degree-decreasing path P from it
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- All in all, $\leq 2 + 2 + 4 + 1 = 9$ colours/l.i. graphs

Third ingredient: Decomposing degenerate graphs and graphs with large minimum degree

Theorem – B., Merker, Thomassen, 2016

Every connected graph of even size decomposes into $H + D$, where $\delta(H) \geq 10^{10}$, and D is an even-size connected $(2 \cdot 10^{10} + 2)$-degenerate graph.

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Theorem – Przybyło, 2016

Every connected graph G with $\delta(G) \geq 10^{10}$ decomposes into 3 l.i. graphs.

We now have everything ☺

Connected graph

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Connected graph

odd size?

Subdivided star

1 l.i. graph

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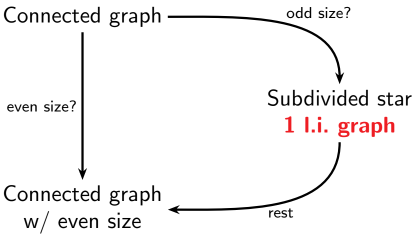

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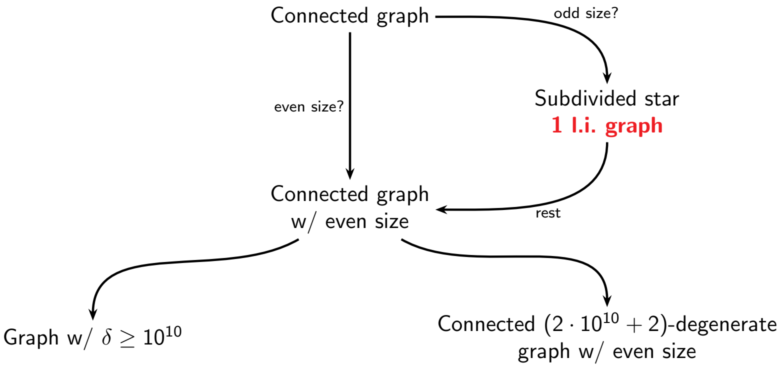

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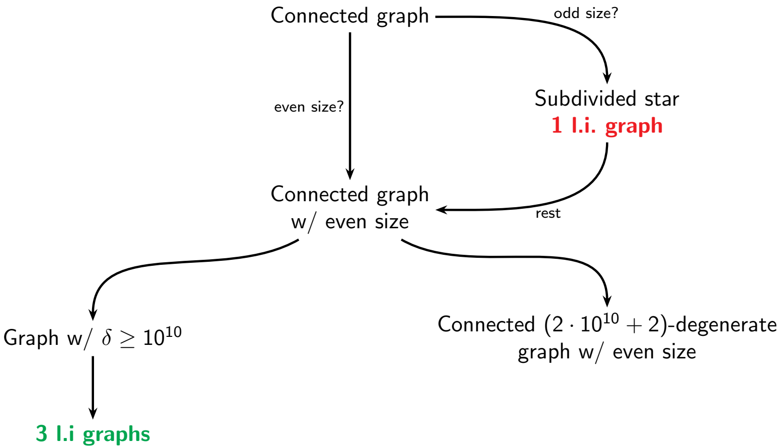

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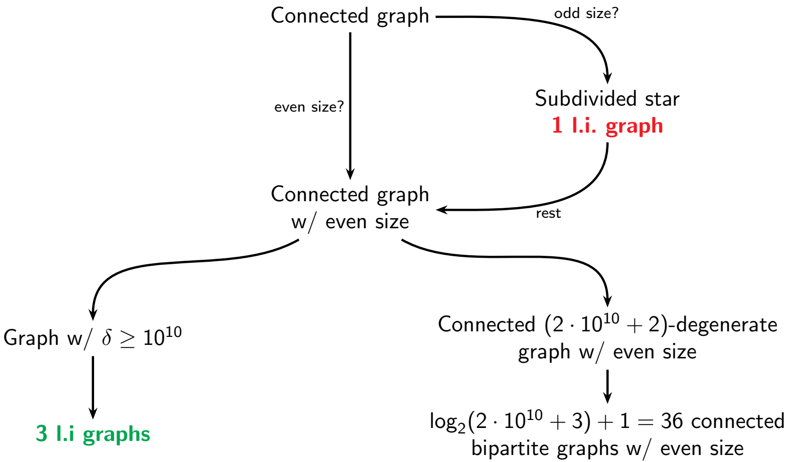

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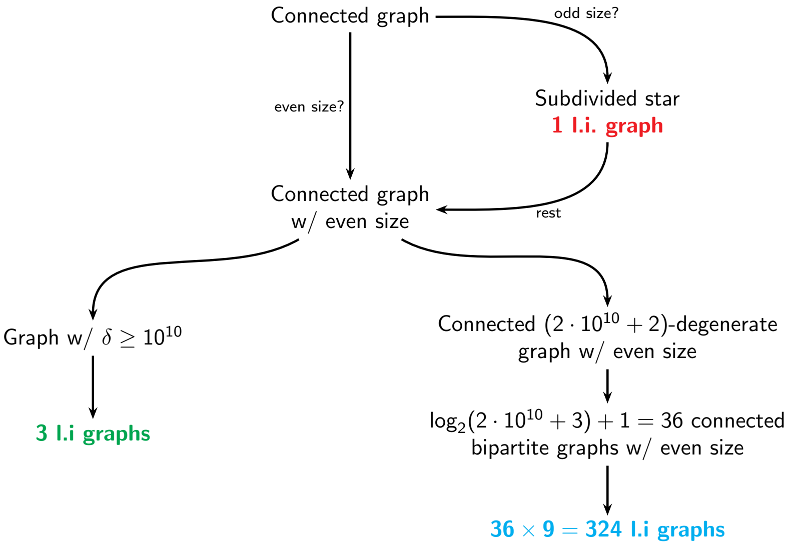

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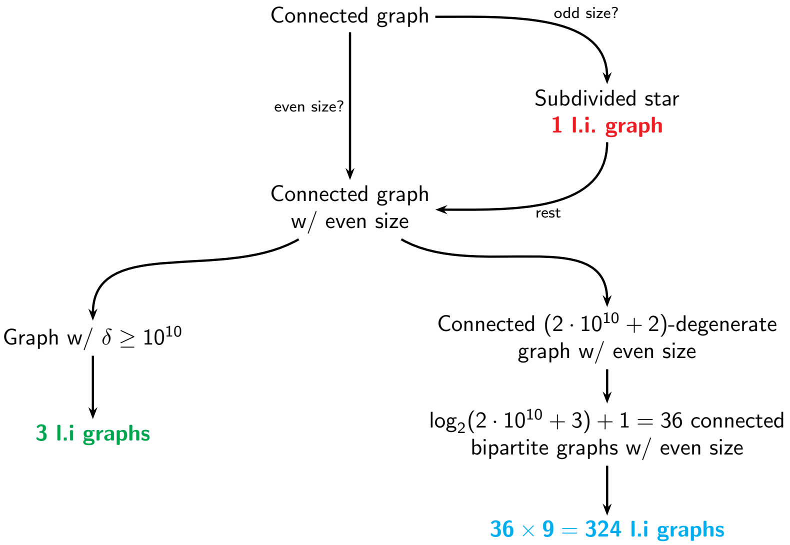

Thus, $\leq 1 + 3 + 324 = 328$ l.i. graphs ☺☺☺

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