The Barát-Thomassen Conjecture

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Introduction

Decomposing graphs

G: (undirected simple) graph.
H: (undirected simple) graph with $|E(H)|$ dividing $|E(G)|$ (implicit).

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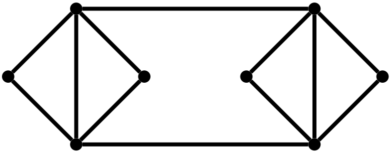

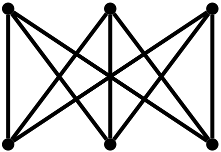

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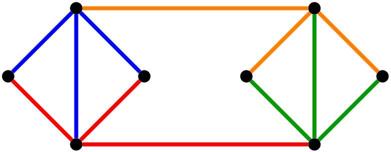

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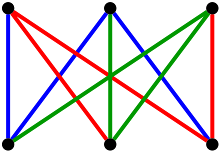

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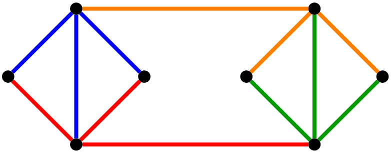

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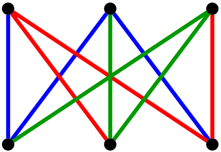

P_3-decomposition

When does G admit H-decompositions?

Tree decompositions

What for H being a tree?

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Theorem [Wilson, 1976]

For every tree T and large enough n, graph K_n admits T-decompositions.

$\Rightarrow$ Intuitively, need large degree + some edge-connectivity (2nd $\Rightarrow$ 1st).

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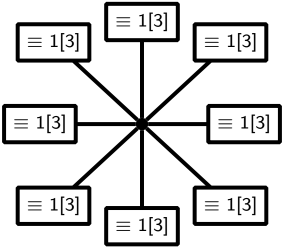
For instance, no P_3-decomposition of:



The Barát-Thomassen Conjecture

Conjecture [Barát, Thomassen, 2006]

For every tree T, there exists k_T such that every k_T-edge-connected graph admits T-decompositions.

General remark:

Large edge-co. $\not\Rightarrow$ H-decompositions (e.g. $H = C_4$: need close cut edges)

Progress towards the conjecture

Was verified for T being:

- a star [Thomassen, 2012],
- the tree with degree sequence $(1, 1, 1, 2, 3)$ [Barát, Gerbner, 2014],
- a bistar of the form $S_{k,k+1}$ [Thomassen, 2014],
- of **diameter at most 4 [Merker, 2017]**,
- among some family of trees with diameter 5 [Merker, 2017],

and...

- the path of length 3 [Thomassen, 2008],
- the path of length 4 [Thomassen, 2008],
- a **path of length 2^k [Thomassen, 2014]**,
- **any path [Botler, Mota, Oshiro, Wakabayashi, 2017]**.

Main result

Theorem [B., Harutyunyan, Le, Merker, Thomassé, 2017]

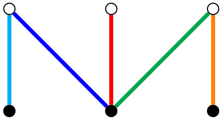
The Barát-Thomassen Conjecture is true.

Please: Do not ask me about k_T ☺.

Proof

Say hello

Our toy T for today:



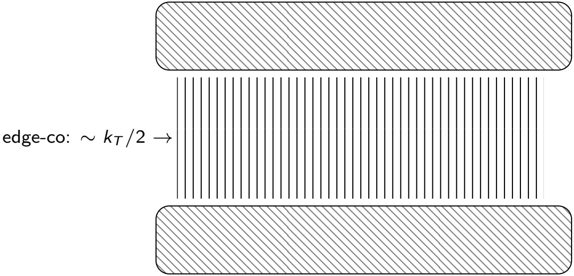
Going bipartite

First tool:

Theorem [Thomassen, 2013]

It is sufficient to prove the conjecture for G bipartite.

Idea: Take a max cut and "clean".



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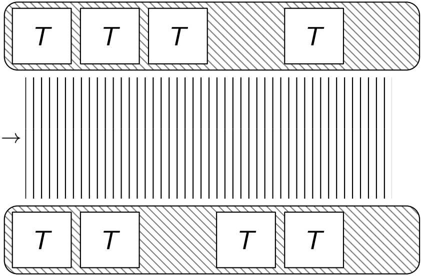
edge-co: $\sim k_T/2 \rightarrow$



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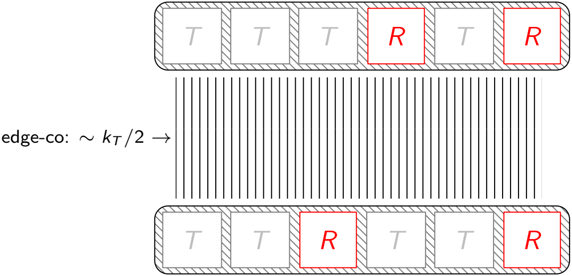

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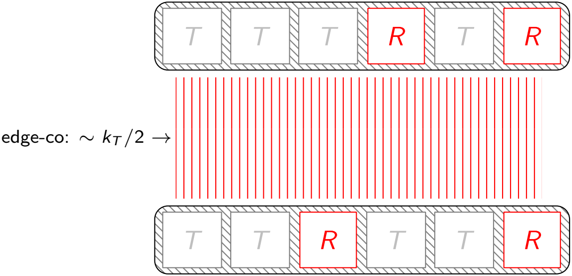

$\Rightarrow$ Use R + cut-edges to make further copies of T.
$|E(T)|$ fixed $\Rightarrow$ constant amount of consumed edge-connectivity. ■

Going bipartite (cont'd)

Theorem [Thomassen, 2013]

It is sufficient to prove the conjecture for $G = (A, B)$ bipartite, with the further assumption that all degrees in A are divisible by $|E(T)|$.

Idea. Decompose G into G_1, G_2 with large edge-connectivity, where the desired property in G_1 (resp. G_2) is fulfilled in A (resp. B).

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2. $\Rightarrow$ Decompose G into G_1, G_2, G_3 with large edge-connectivity.

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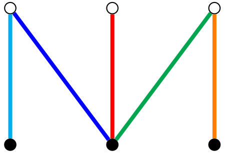
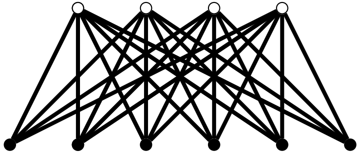
4. Add all arcs from A to B to G_1, to G_2 otherwise. ∎

Decomposition strategy

G, T bipartite $\Rightarrow$ Make the bipartitions coincide:





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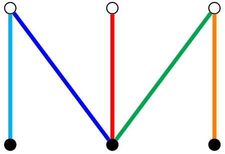

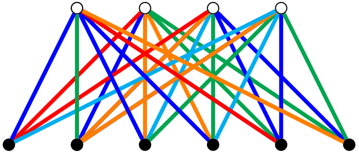

Strategy:

1. Edge-colour G with { / , / , / , / , / };
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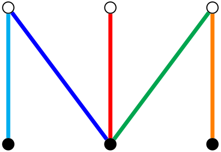

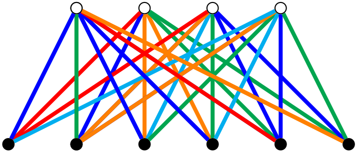

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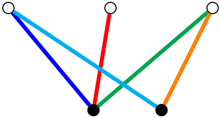
Problems ☹ :

1. # of /'s , /'s , /'s , /'s and /'s should locally be the same.
2. We do not necessarily get a copy **isomorphic** to T:



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$v \in V(G)$ and $t \in V(T)$ **compatible** $=$ Same side of the bipartitions.

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What will save us:

Theorem [Merker, 2017]

If $G = (A, B)$ is a bipartite graph with

- sufficiently large edge-connectivity, and
- all degrees in A are divisible by $|E(T)|$,

$\Rightarrow$ T-equitable edge-colouring where all coloured degrees are "huge".

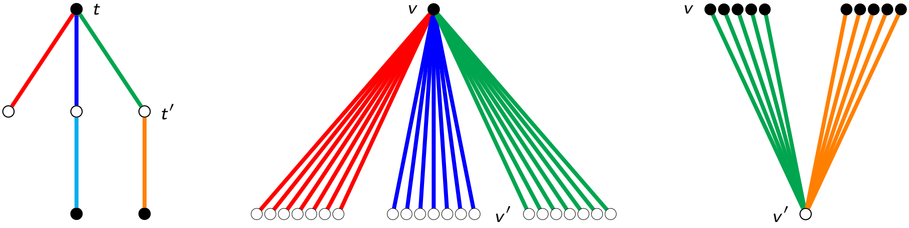
$\Rightarrow$ May assume G is edge-coloured in a T-equitable way.

Building a decomposition

Locally, "palettes" of colours are good, now ☺ .

Construct copies of T:

1. For each $v \in G$ that can play the role of $t \in T$:
   - choose one edge of each colour;
   - create a star centred at v.
2. Identify stars to create copies.



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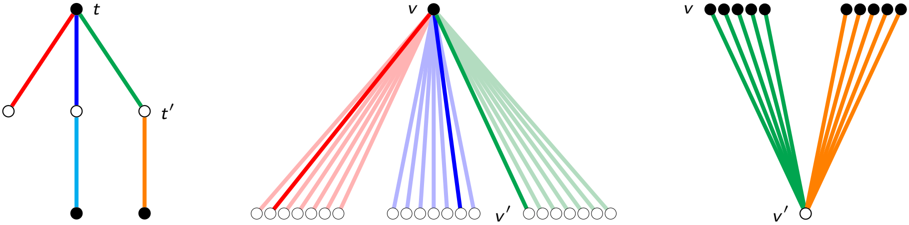

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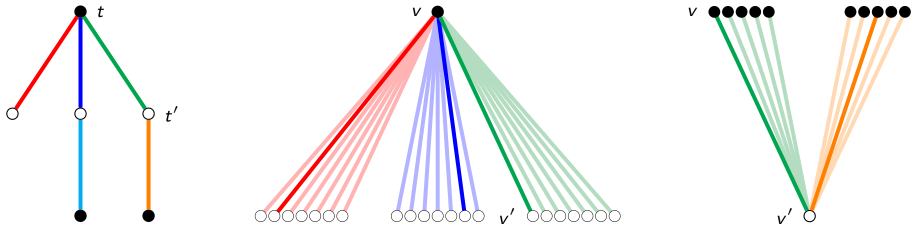

Dealing with Issue 2

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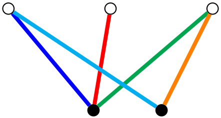

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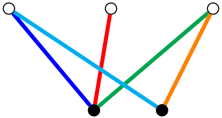
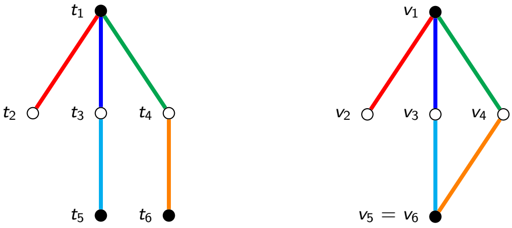


$\Rightarrow$ Collection $\mathcal{H} := \mathcal{G} \cup \mathcal{B}$, where $\mathcal{G}$ (resp. $\mathcal{B}$) contains "real" (resp. "bad") copies.

$\mathcal{G}$ will be used to "repair" $\mathcal{B}$.

Repairing process

Let $B \in \mathcal{B}$, and label vertices following a BFS.



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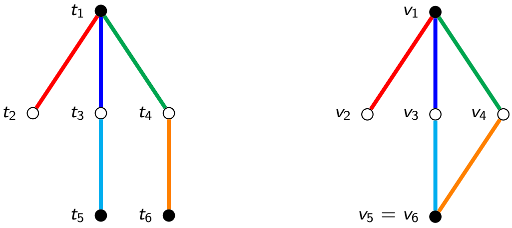

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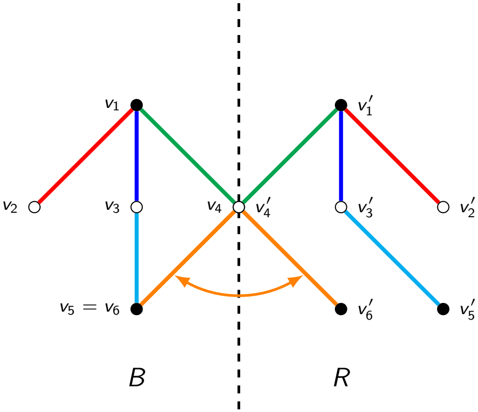
Repairing process:

1. Pick $R \in \mathcal{G}$ s.t. B and R intersect only intersect in v_4; and
2. "Switch" the subgraph "rooted" at the edge $v_4 v_6$.

Illustration



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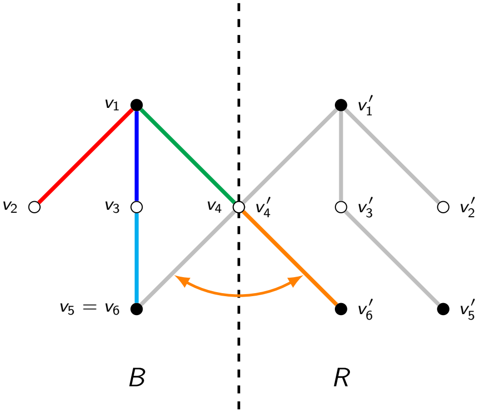

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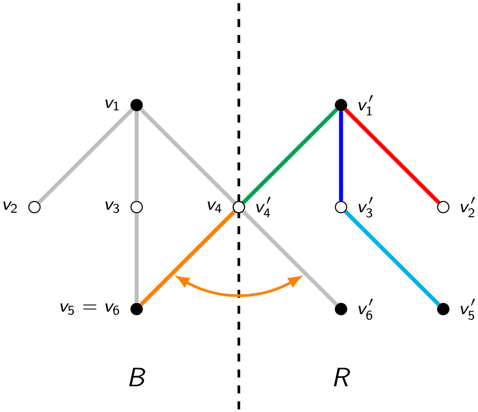
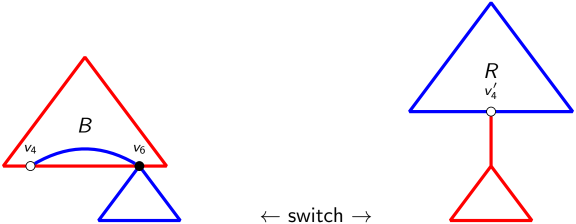


On the repairing operation

Schematized:



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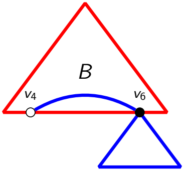
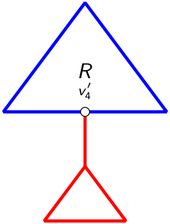


← switch →



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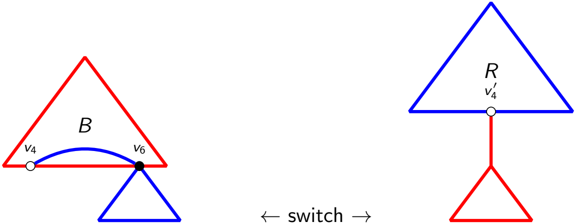

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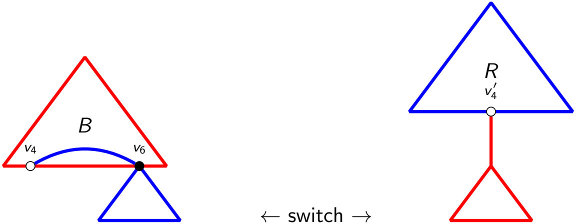

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2. Then, those where the edge simulating $t_1 t_3$ is problematic;
3. etc.

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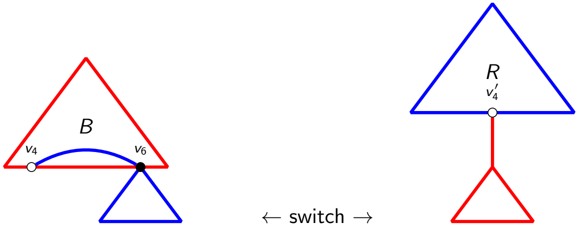

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$|\mathcal{G}| \gg |\mathcal{B}|$ (+ intersection property) $\Rightarrow$ Repair everything.

Proof summary

Assumption: G is edge-coloured in a T-equitable way + Large coloured degrees.

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$\Rightarrow$ Because

1) $|E(T)|$ is fixed, and
2) the coloured degrees are arbitrarily large,

such an $\mathcal{H}$ exists with non-zero probability.

Probabilistic tools

Building a decomposition

Construct copies of T **randomly**:

1. For each $v \in G$ that can play the role of $t \in T$:
    - choose one edge of each colour;
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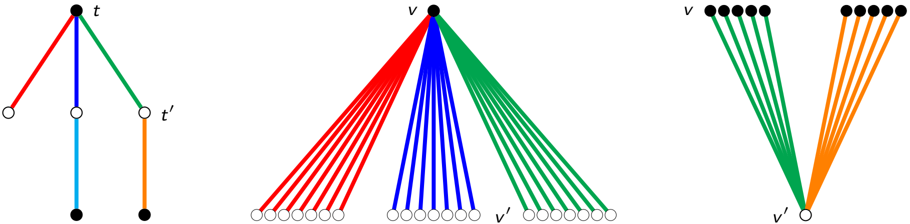

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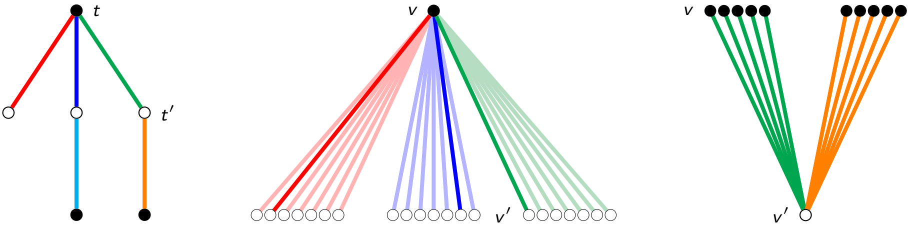

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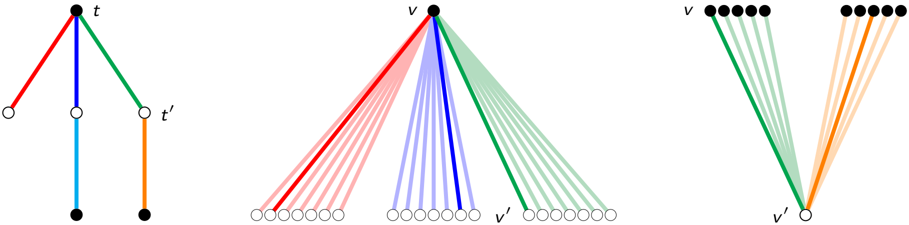

McDiarmid's result

Random variables involved:

$X_v(t_i, t_j) := \#$ of bad copies with root v, and t_i, t_j played by a same vertex.

$\Rightarrow$ Expect such $X_v(t_i, t_j)$'s to be quite small (due to the degrees):

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(simplified) McDiarmid's Inequality

Let X be a non-negative random variable, determined by m independent random permutations $\Pi_1, ..., \Pi_m$ satisfying, for some $d, r > 0$:

1. interchanging two elements in any Π_i can affect X by at most d;
2. for any s, if $X \geq s$ then there is a set of at most rs choices whose outcomes certify that $X \geq s$.

Then, for any $0 \leq t \leq \mathbb{E}[X]$,

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > t + 60d\sqrt{r\mathbb{E}[X]}\right] \leq 4e^{-\frac{t^2}{8d^2 r\mathbb{E}[X]}}.$$

McDiarmid's result (cont'd)

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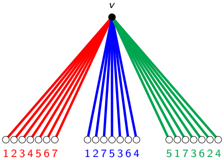

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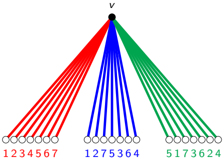

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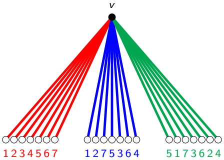

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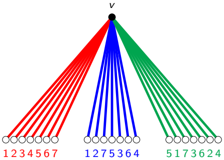

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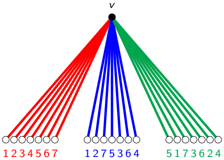

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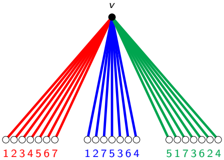

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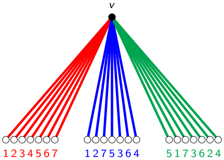

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+ For compatible $v \in G$ and $t \in T$, unlikely that two copies where v plays the role of t have another common vertex (similar reasoning).

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