Strong edge-colouring of sparse planar graphs

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G: undirected simple graph
Δ: maximum degree of G

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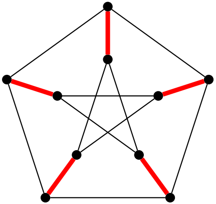

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$\chi'(G)$: $\min\{k : G \text{ has a proper } k\text{-edge-colouring}\}$

Theorem [Vizing, 1964]

Either $\chi'(G) = \Delta$ (Class 1) or $\chi'(G) = \Delta + 1$ (Class 2).

Theorem [Holyer, 1981]

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better complexity in case G is bipartite

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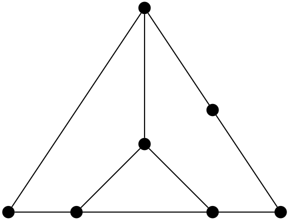

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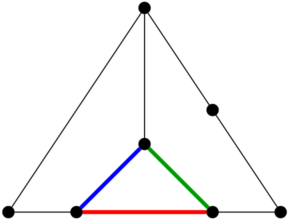

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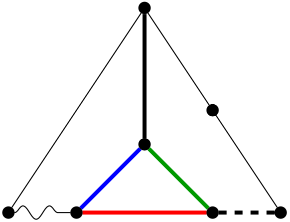

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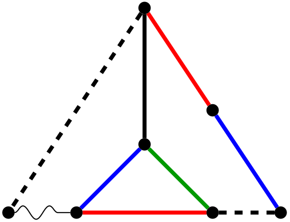

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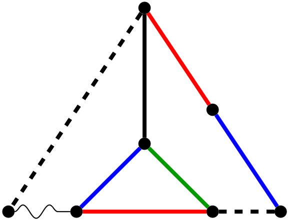

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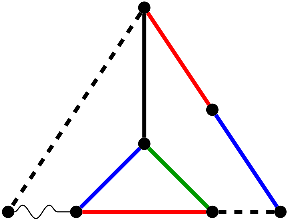

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= proper vertex-colouring of $L(G)^2$

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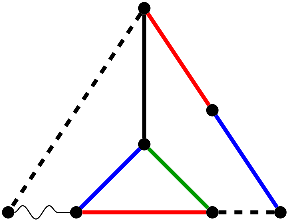

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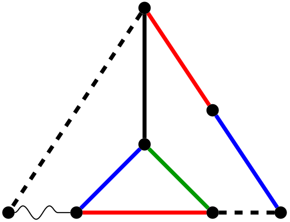
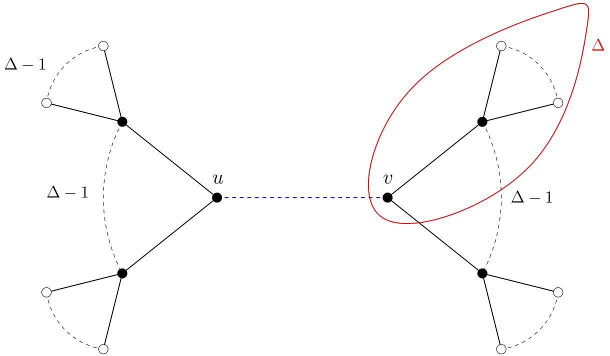


$\chi'_s(G)$: $\min\{k : G$ has a strong k-edge-colouring$\}$
$\chi'_s(G) = \chi(L(G)^2)$

Upper bounds

$\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1$ (counting)



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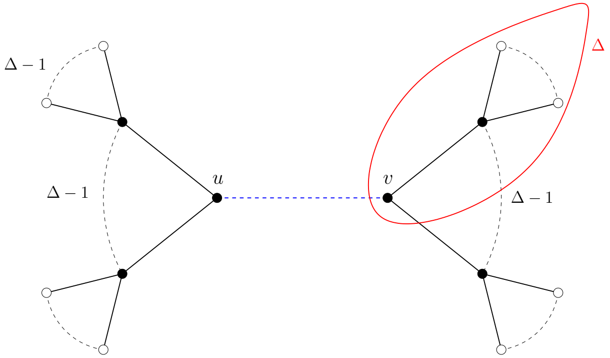

Theorem [Molloy and Reed, 1997]

If Δ is large enough, then $\chi'_s(G) \leq 1.998\Delta^2$.

Conjecture [Erdős and Nešetřil, 1985]

We have $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 \text{ for } \Delta \text{ even, and} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \text{ otherwise.} \end{cases}$

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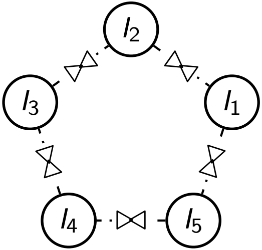
verified for $\Delta \leq 3$ (Andersen, 1992, Horák *et al.*, 1993)

tightness: consider C_5^Δ, where



- every I_j is an independent set
- if $\Delta = 2k$, then $|I_j| = k$
- if $\Delta = 2k+1$, then $|I_1| = |I_2| = |I_3| = k$ and $|I_4| = |I_5| = k+1$

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Theorem [Chung *et al.*, 1990]

If G has no induced $2K_2$, then

$$|E(G)| \leq \begin{cases} \frac{5}{4}\Delta^2 \text{ for } \Delta \text{ even, and} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \text{ otherwise.} \end{cases}$$

Besides, these upper bounds are reached if and only if $G = C_5^\Delta$.

Other graph classes

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Theorem [*alii*, 2012+]

If G is k-degenerate, then $\chi'_s(G) \leq (4k-2)\Delta - 2k^2 + \mathcal{O}(k)$.

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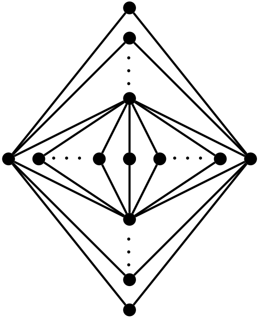

$$\chi'_s(G) = 4\Delta - 4$$

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If G is outerplanar, then $\chi'_s(G) \leq 3\Delta - 3$. This bound is tight.

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g: girth of G

Theorem [Hudák *et al.*, 2013]

If G is planar with $g \geq 6$, then $\chi'_s(G) \leq 3\Delta + 5$.

some such graphs need $2.4\Delta + c$ colours

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Theorem [B., Harutyunyan, Hocquard and Valicov, 2013+]

If G is planar with $g \geq 6$, then $\chi'_s(G) \leq 3\Delta + 1$.

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Proof outline

H: minimal (vertices+edges) counterexample

1. structural properties of H
2. discharging procedure
   2.1 weight function ω: for every $x \in V(H)$, set $\omega(x) = 2d(x) - 6$ such that $\sum_{x \in V(H)} \omega(x) < 0$
   2.2 discharging rules
   2.3 new weight function ω^* such that $\sum_{x \in V(H)} \omega(x) = \sum_{x \in V(H)} \omega^*(x)$
3. using 1., we get to the contradiction

$$0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0$$

H cannot exist

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proved for $\Delta = 7$ (Sanders and Zhao, 2001)

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Question

If $\Delta = g = 4$, then can G be Class 2?

Summary

	$\Delta \geq 7$	$\Delta \in \{5, 6\}$	$\Delta = 4$	$\Delta = 3$
no girth restriction	4Δ	$\mathbf{4\Delta + 4}$	$\mathbf{4\Delta + 4}$	$3\Delta + 1$
$g \geq 4$	4Δ	4Δ	$\mathbf{4\Delta + 4}$	$3\Delta + 1$
$g \geq 5$	4Δ	4Δ	4Δ	$3\Delta + 1$
$g \geq 6$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$	3Δ
$g \geq 7$	3Δ	3Δ	3Δ	3Δ

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$g \geq 6$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$	3Δ
$g \geq 7$	3Δ	3Δ	3Δ	3Δ

Thank you for your attention!