Edge-partitioning a graph into paths: beyond the Barát-Thomassen conjecture

Julien Bensmail, Ararat Harutyunyan,
Tien-Nam Le and Stéphan Thomassé

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GT 2015, Denmark
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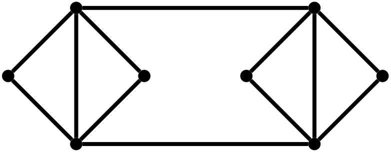
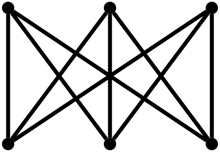
Decomposition of graphs

T-decomposition: **edge-partition** into **copies of T**.





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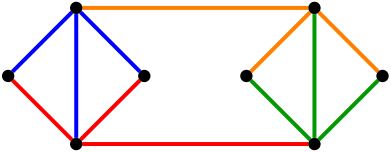

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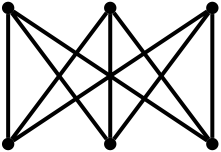

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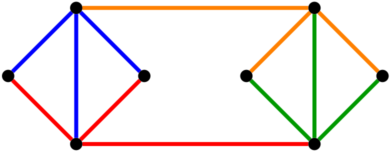

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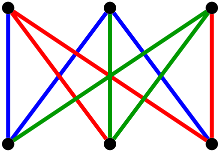

P_3-decomposition

Barát-Thomassen conjecture

Conjecture [Barát, Thomassen – 2006]

For every fixed tree T, there exists a positive constant c_T such that every c_T-edge-connected graph with size divisible by $|E(T)|$ admits a T-decomposition.

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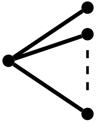
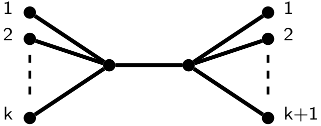


stars
[Thomassen – 2012]



$(k, k+1)$-bistars
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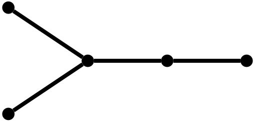

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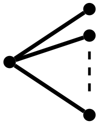

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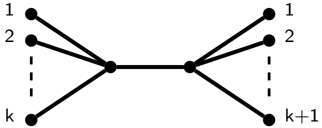

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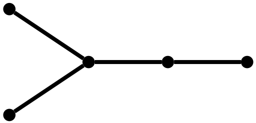

of deg. sequence $(1, 1, 1, 2, 3)$
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... and actually whenever $\mathrm{diam}(T) \leq 4$ [Merker – 2015+].

Barát-Thomassen conjecture for paths

Also true for $T = P_\ell$ when (chronological order):

- $\ell \in \{3, 4\}$ [Thomassen – 2008],
- $\ell = 2^k$ for any k [Thomassen – 2013],
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Shorter and “easier” proof of the path case?

Our results

Contribution: **New proof** of the path case under **weaker assumptions**.

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Mild edge-connectivity is sufficient provided minimum degree is large enough.

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For every $\ell \geq 1$, every 24-edge-connected graph admits a P_ℓ-decomposition (+1 smaller path) provided its minimum degree is large enough.

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Even less edge-connectivity needed for eulerian graphs.

Theorem [B., Harutyunyan, Le, Thomassé – 2015+]

For every $\ell \geq 1$, every 4-edge-connected **eulerian** graph admits a P_ℓ-decomposition (+1 smaller path) provided its minimum degree is large enough.

Note: size condition is dropped.

About tightness

The following is optimal:

- 3-edge-connectivity for non-eulerian graphs,
- 2-edge-connectivity for eulerian graphs.

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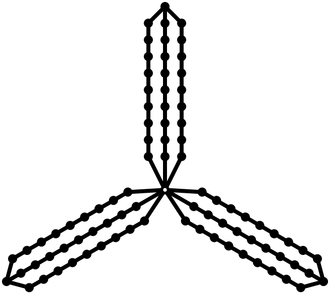
Note: 2-edge-connectivity does not suffice for the first item; *e.g.* for



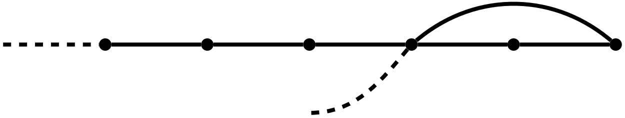
... and make δ increase with preserving non P_9-decomposability.

Outline of the proof

Theorem [B., Harutyunyan, Le, Thomassé]

For every $\ell \geq 1$, every 24-edge-connected graph admits a P_ℓ-decomposition provided its minimum degree is large enough.

Proof ideas. Assume G has an euler tour Γ, and pick consecutive P_ℓ's.



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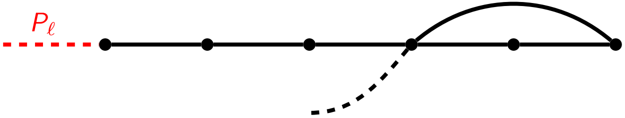

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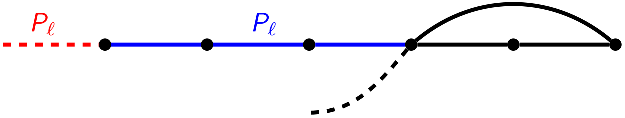

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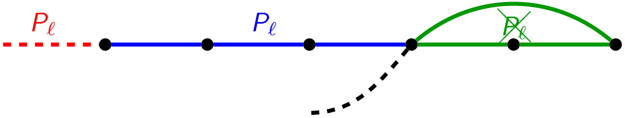
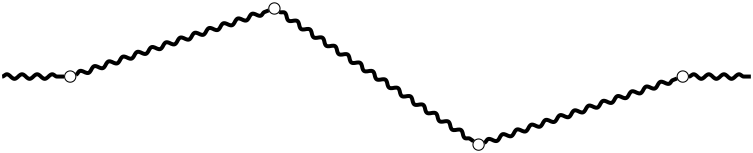


Problem: Γ may be of small girth.

Notion of path-graph

Solution: Decompose G into paths of length at least ℓ (*i.e.* express G as an **($\geq \ell$)-path-graph** H), and decompose an euler tour Γ going through the paths.



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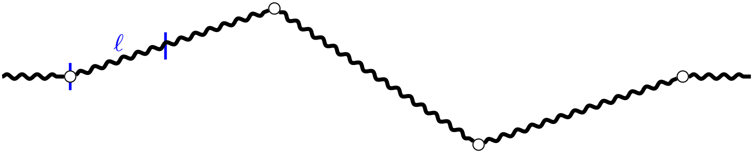

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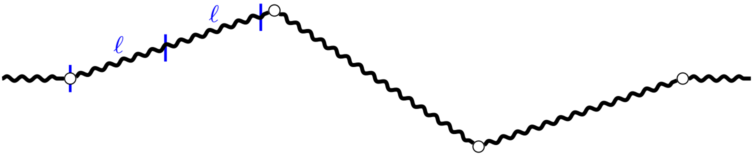

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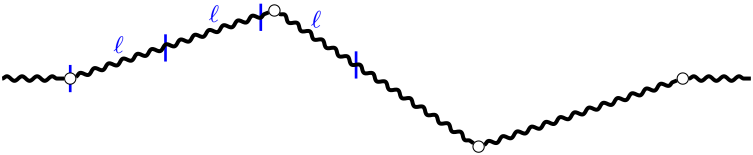

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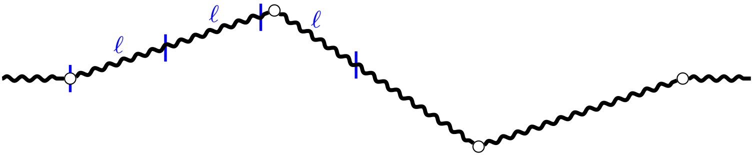

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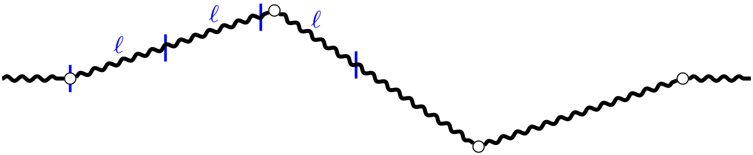

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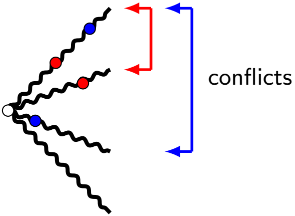

$$\mathsf{conf}(v) := \frac{\max_{w \neq v} \left| \{ P \in \mathcal{P}_H(v) \, : \, w \in P \} \right|}{d_H(v)} \qquad \mathrm{conf}(H) := \max_v \mathsf{conf}(v)$$

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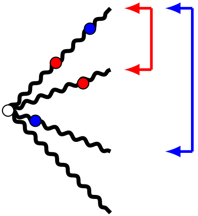

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Theorem [Jackson – 1993]

Every eulerian path-graph H with $\mathrm{conf}(H) \leq 1/2$ has a conflictless euler tour.

From graphs to path-graphs

Graph with large $\delta \Rightarrow (\geq \ell)$-path-graph with **arbitrarily low conflicts**.

Theorem [B., Harutyunyan, Le, Thomassé – 2015+]

For every $\ell \geq 1$, every graph with large enough minimum degree can be expressed as an $(\geq \ell)$-path-graph H with arbitrarily low $\mathrm{conf}(H)$.

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Problem remaining: Ensuring Eulerianity of H??

Solution: Extract subgraphs of G that will be used to "repair" the *connectivity* and the *degrees* of H (if necessary).

Ensuring Eulerianity of a path-graph

Cautious: Adding paths to H may increase $\mathrm{conf}(H)$ too much.

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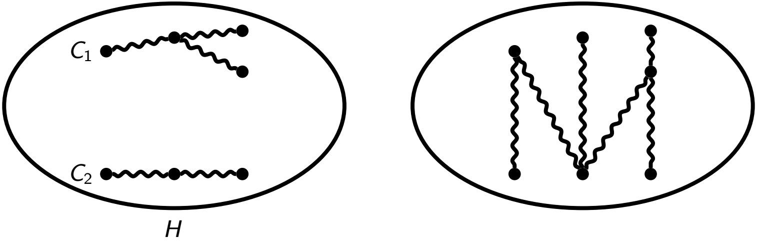

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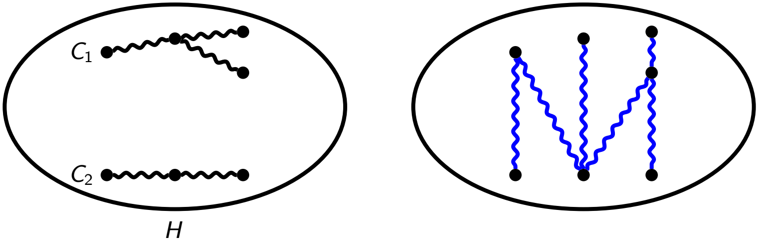

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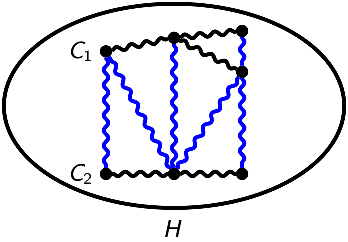

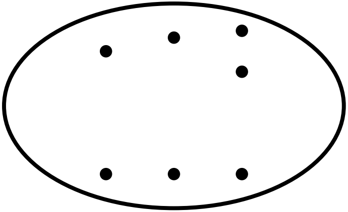

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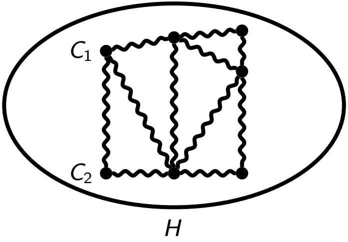
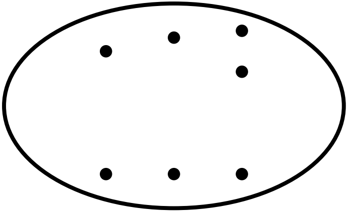
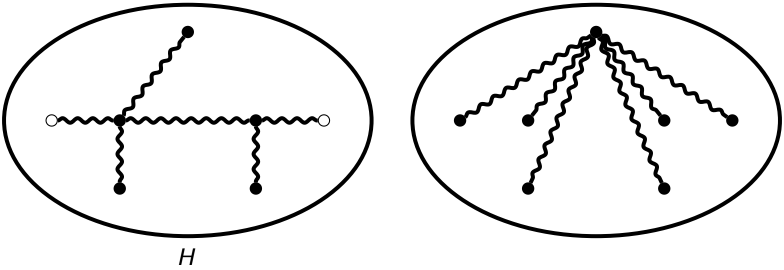




Making all vertices being of even degree

Even degrees? $\Rightarrow$ Add **paths** to H **joining vertices of odd degree**.

Just need an $(\geq \ell)$-path-graph tree with bounded maximum degree.



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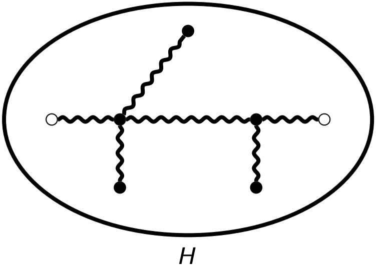

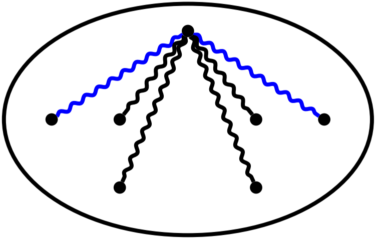

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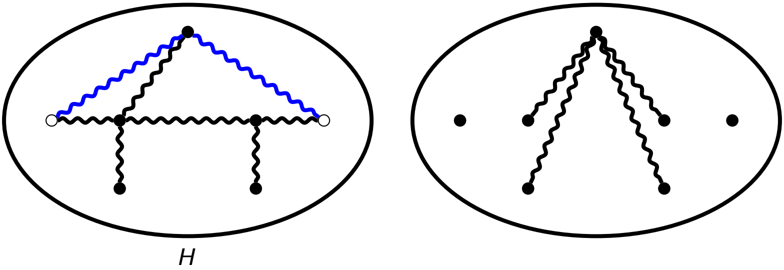

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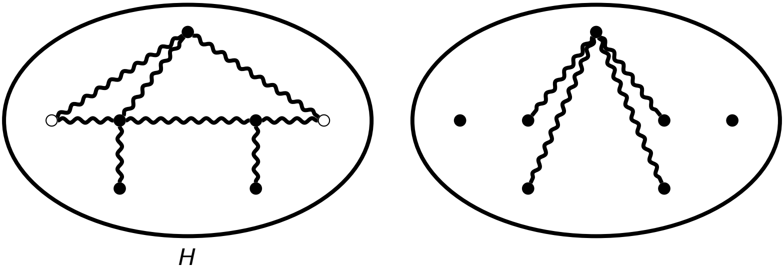

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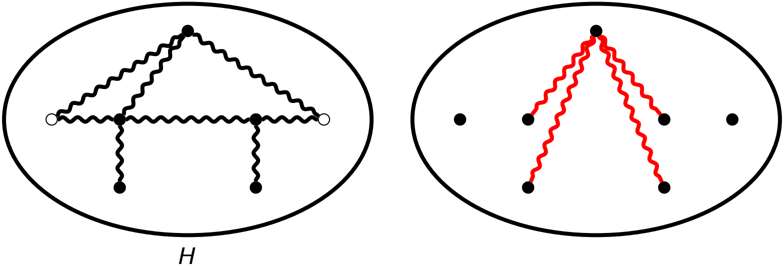

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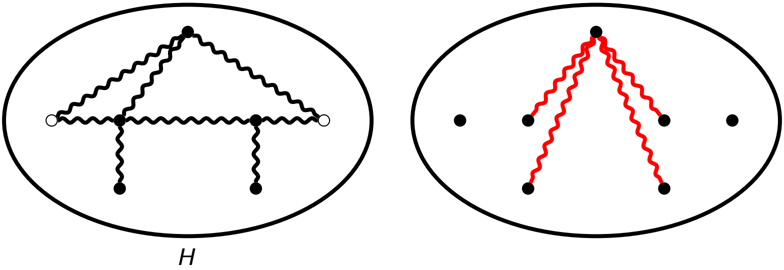

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Solution: Make sure these paths have length multiple of ℓ.

Finding $(\ell, 2\ell)$-trees with bounded maximum degree

$(\ell, 2\ell)$-**tree:** path-graph tree whose paths have length ℓ or 2ℓ.

Repairing connectivity and degrees $\Rightarrow$ Two $(\ell, 2\ell)$-**trees** with **bounded** Δ.

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Existence under mild requirements.

Theorem [B., Harutyunyan, Le, Thomassé]

For every $\ell \geq 1$, given a 2-edge-connected graph and a large enough disjoint source of degree, one can obtain an $(\ell, 2\ell)$-tree with maximum degree bounded by a function of ℓ **only**.

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Proof idea:

- 2-edge-connected $\Rightarrow$ subcubic $(1, 2)$-tree.
- $(1, k)$-tree with bounded Δ + degree $\Rightarrow$ $(1, k+1)$-tree with bounded Δ.
- $(1, k+1)$-tree with bounded Δ + degree $\Rightarrow$ $(k, 2k)$-tree with bounded Δ.

Final picture

24-edge-connected
large δ

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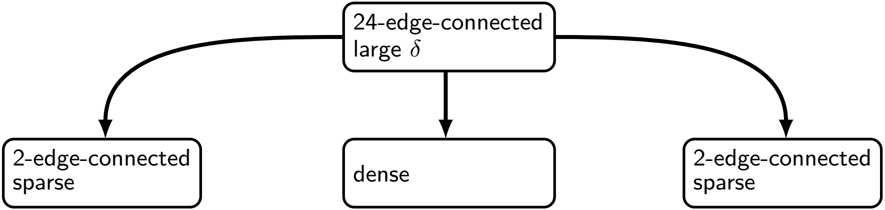

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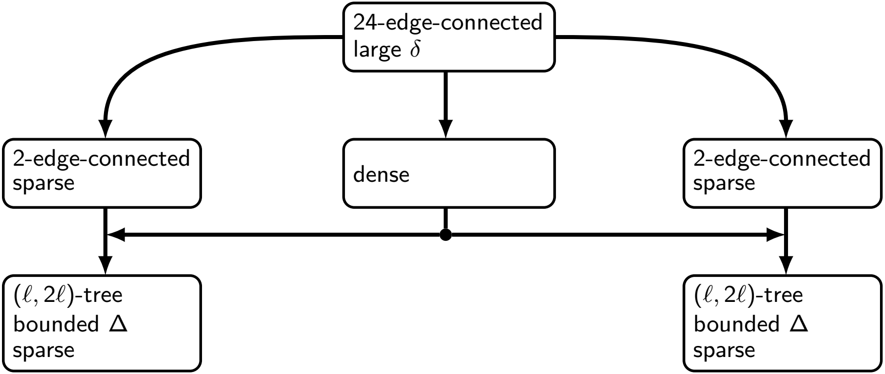

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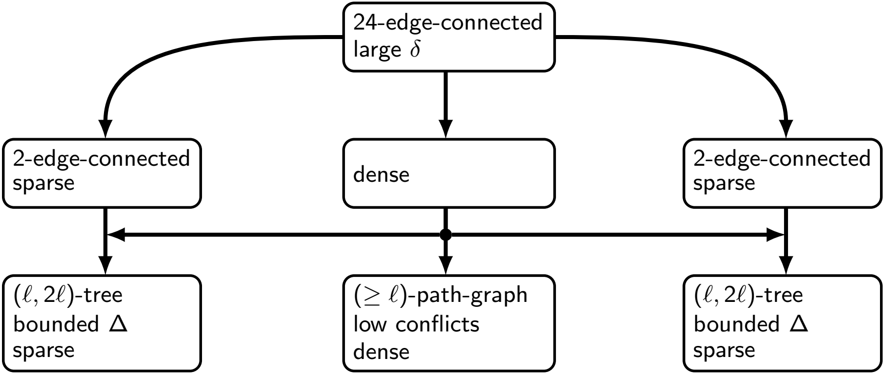
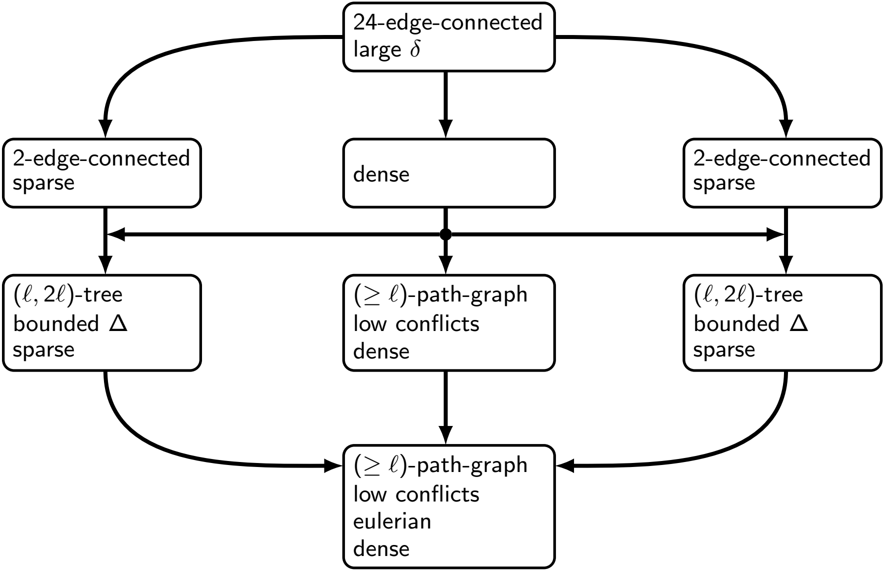
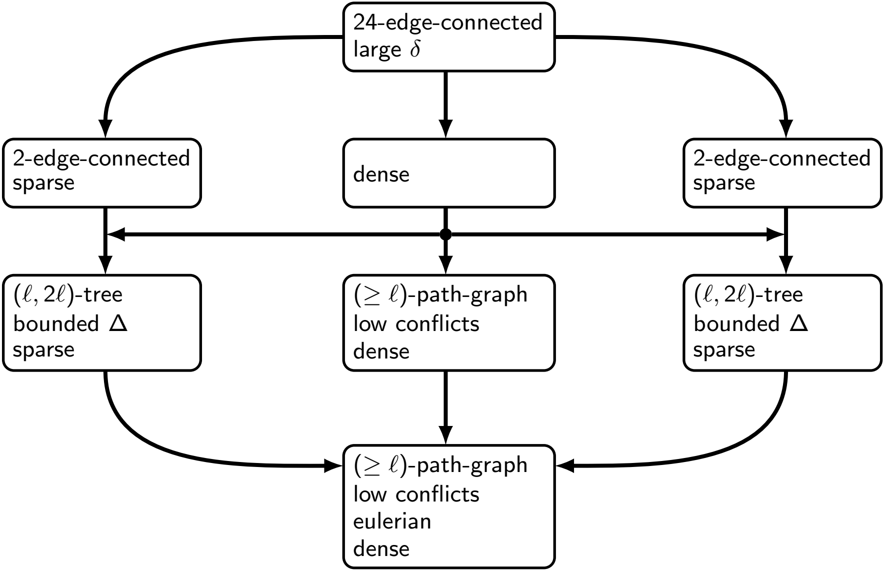


Final picture



Final picture



$\Rightarrow$ conflictless euler tour $\Rightarrow$ P_ℓ-decomposition.

Concluding remarks and questions

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Thank you for your attention.