Decomposing oriented graphs into six locally irregular subgraphs

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LaBRI - Université de Bordeaux
Talence, France

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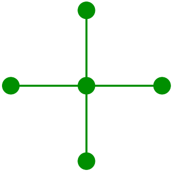
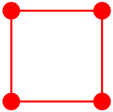
Recall last episode...

G is *locally irregular* if its adjacent vertices have distinct degrees.





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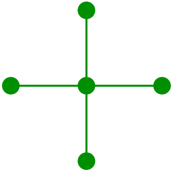
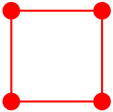




If G is not locally irregular, then we would like to edge-partition G to get the minimum number of edge-disjoint locally irregular subgraphs.

Equivalently, we look for the least number $\chi'_{irr}(G)$ of an edge-colouring of G whose each colour class induces a locally irregular subgraph.

Conjecture (Baudon, B., Przybyło, Woźniak - 2013+)

We have $\chi'_{irr}(G) \leq 3$ unless G is indecomposable.

Part 1: On locally irregular oriented graphs

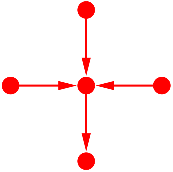
Part 2: We have $\chi'_{irr}(\overrightarrow{G}) \leq 6$ for every $\overrightarrow{G}$

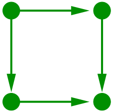
Part 3: Deciding whether $\chi'_{irr}(\overrightarrow{G}) \leq 2$ is NP-complete

Part 4: Conclusion and open questions

An oriented version of the same problem

$\overrightarrow{G}$ is *locally irregular* if its adjacent vertices have distinct *outdegrees*.





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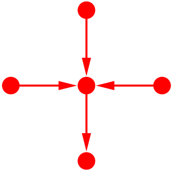

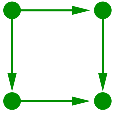

Again, if $\overrightarrow{G}$ is not locally irregular, then we want to partition its arcs into the minimum number $\chi'_{irr}(\overrightarrow{G})$ of parts inducing locally irregular subgraphs.

Conjecture (B., Renault - 2013+)

We have $\chi'_{irr}(\overrightarrow{G}) \leq 3$ for every oriented graph $\overrightarrow{G}$.

We support this conjecture by showing it to hold when $\chi(und(\overrightarrow{G})) \leq 3$, and when $\overrightarrow{G}$ is acyclic or some Cartesian product of two oriented graphs.

Oriented graphs underlaid by 3-colourable graphs

Theorem (B., Renault - 2013+)

We have $\chi'_{irr}(\overrightarrow{G}) \leq \chi(und(\overrightarrow{G}))$.

Proof. Let $k = \chi(und(\overrightarrow{G}))$. Consider a proper k-vertex-colouring

$$V(und(\overrightarrow{G})) = V_1 \cup V_2 \cup ... \cup V_k$$

of $und(\overrightarrow{G})$, and for every $v \in V_i$, colour i all arcs outgoing from v. Then all subgraphs of $\overrightarrow{G}$ induced by the obtained arc-colouring are locally irregular. ∎

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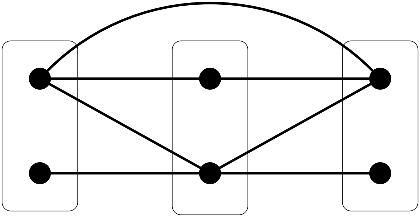

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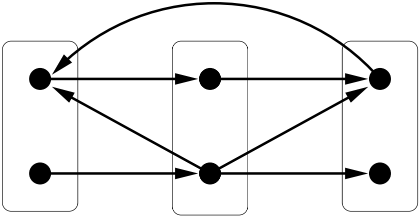

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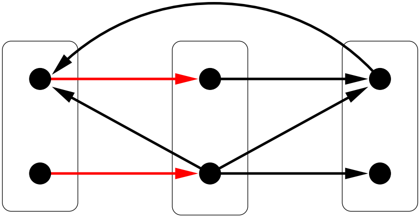

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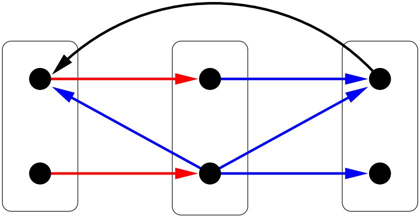

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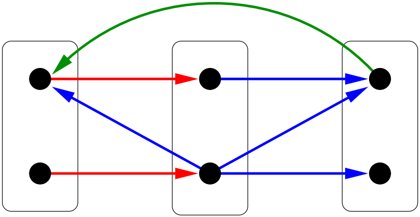

Cartesian products of "decomposable" oriented graphs

Theorem (B., Renault - 2013+)

We have $\chi'_{irr}(\overrightarrow{G} \square \overrightarrow{H}) \leq \max\{\chi'_{irr}(\overrightarrow{G}), \chi'_{irr}(\overrightarrow{H})\}$.

Proof. Just "combine" a locally irregular $\chi'_{irr}(\overrightarrow{G})$-arc-colouring of $\overrightarrow{G}$ and a locally irregular $\chi'_{irr}(\overrightarrow{H})$-arc-colouring of $\overrightarrow{H}$. ■

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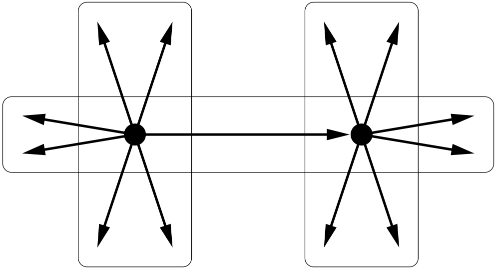

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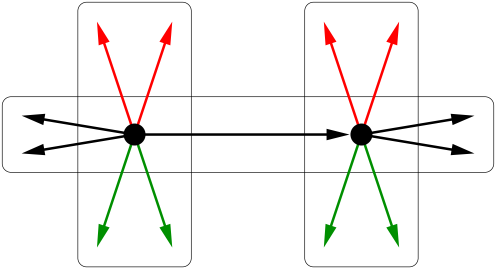

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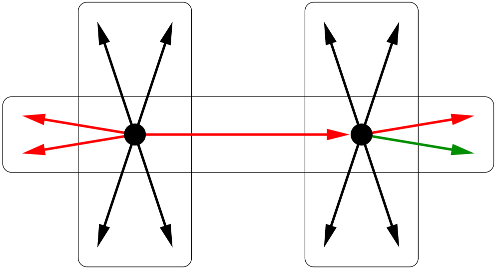

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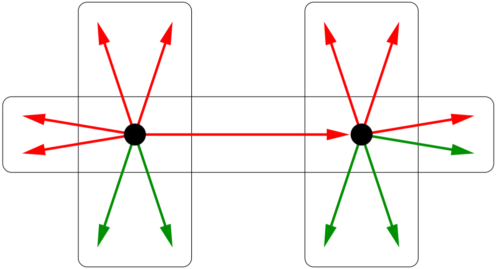

Acyclic oriented graphs

Theorem (B., Renault - 2013+)

We have $\chi'_{irr}(\overrightarrow{G}) \leq 3$ whenever $\overrightarrow{G}$ is acyclic.

Proof. We prove a stronger statement, namely that every acyclic oriented graph admits a locally irregular 3-arc-colouring such that at most two colours are used at each vertex.

This is proved by induction on the size of $\overrightarrow{G}$. As a base case, note that a single arc can be assigned any colour. Now consider the inductive step.

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Because $\overrightarrow{G}$ is acyclic, there is some vertex v with indegree 0. By removing v off $\overrightarrow{G}$, we get a smaller acyclic oriented graph which admits a satisfying locally irregular 3-arc-colouring ϕ according to the induction hypothesis.

We extend ϕ to $\overrightarrow{G}$, i.e. to the arcs outgoing from v.

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Theorem (B., Renault - 2013+)

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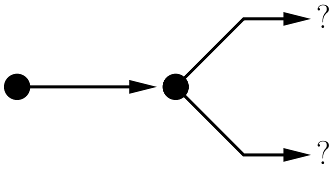
Proof. Suppose $d^+(v) = 1$.



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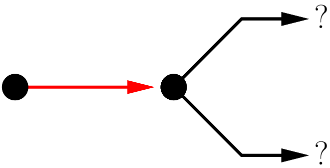

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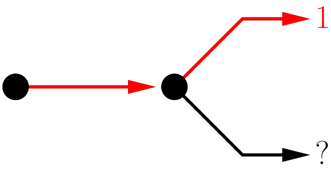

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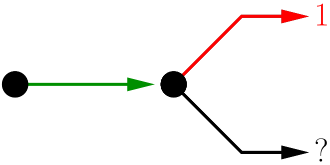

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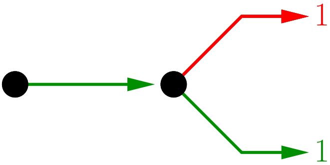

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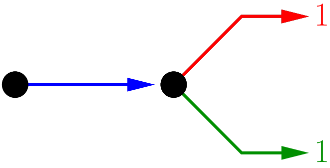

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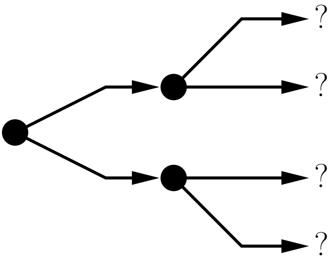

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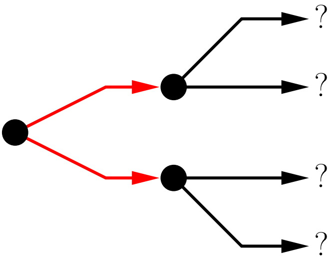

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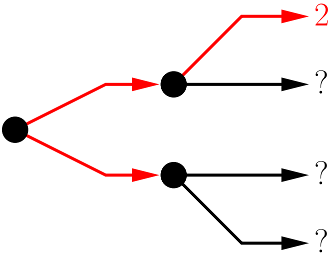

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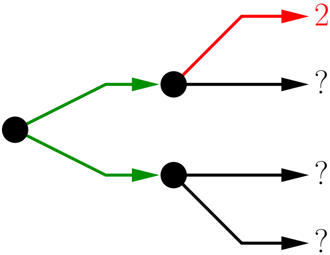

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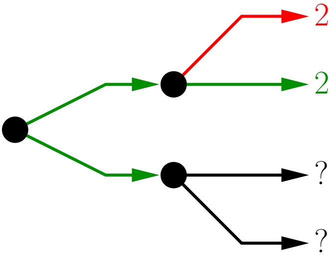

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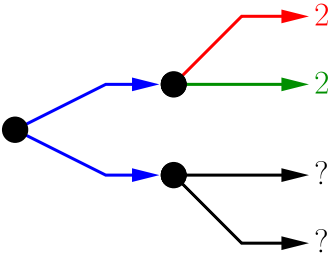

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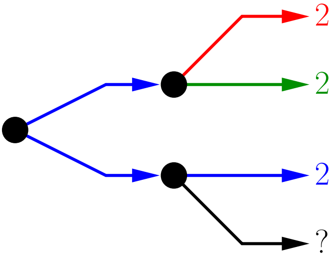

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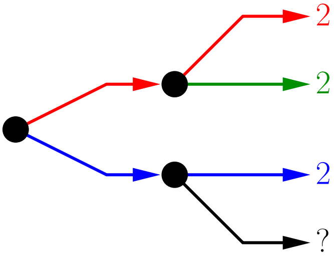

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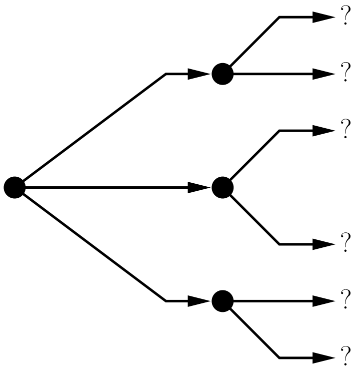

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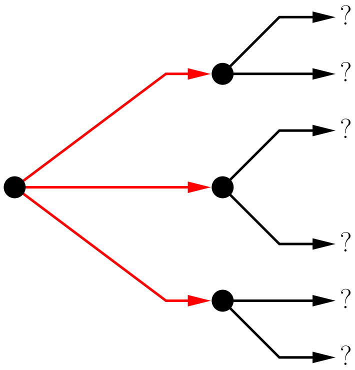

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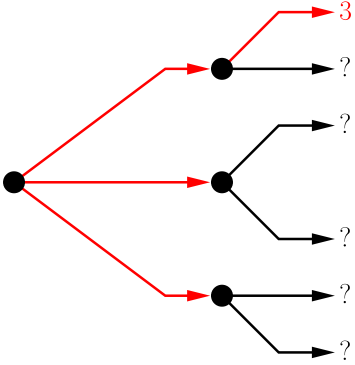

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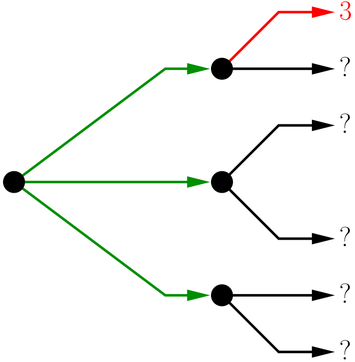

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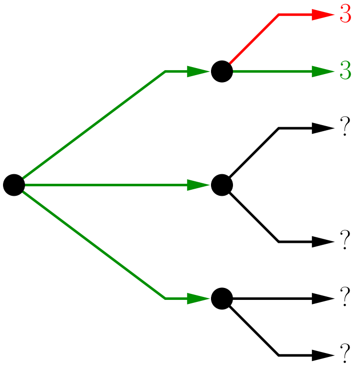

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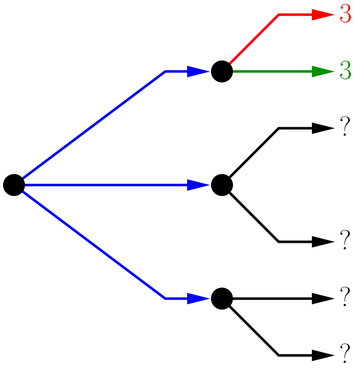

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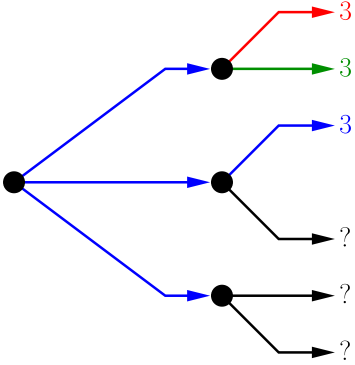

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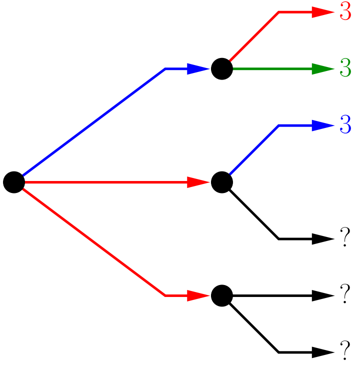

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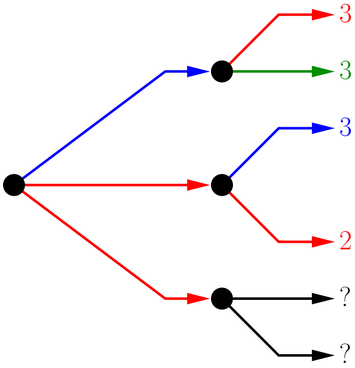

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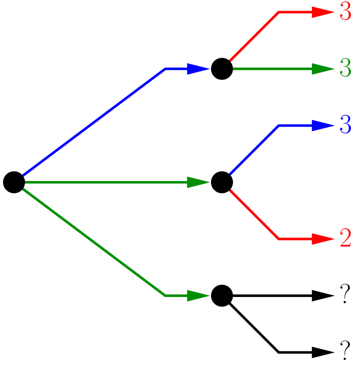

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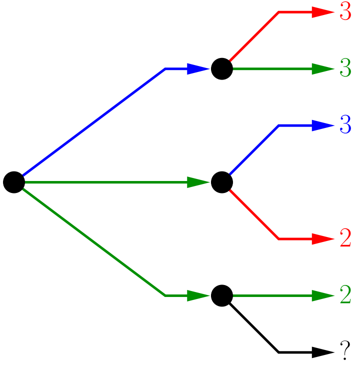

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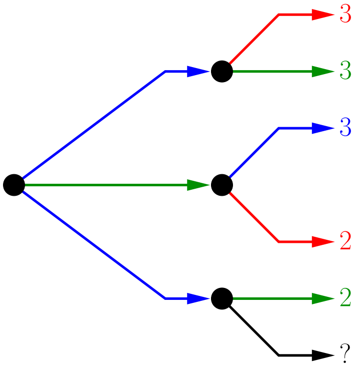

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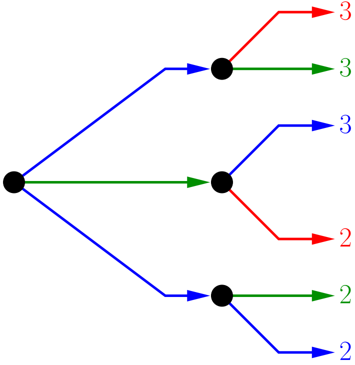

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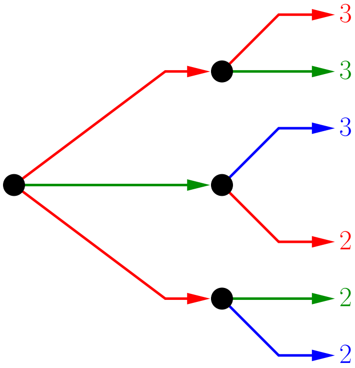

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Proof. This colouring strategy generalizes to every value of $d^+(v) \geq 6$ (smaller values of $d^+(v)$ can be checked separately).

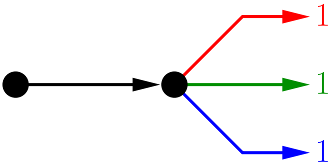
First, try out to colour the arcs outgoing from v in every possible way (but using at most two colours). If one attempt fails, then we may reveal what is some coloured outdegree of one vertex neighbouring v. By leading the process conveniently, we can make sure to discover one coloured outdegree per step.

If none attempt succeeds, then we reveal what are all the coloured outdegrees of the vertices neighbouring v. With a smart strategy, and because at most two colours are used at each vertex neighbouring v, all vertices neighbouring v are revealed to have coloured outdegrees at least $\lceil \frac{d^+(v)}{2} \rceil$.

We can then colour the arcs outgoing from v with two colours so that v has coloured outdegrees $\lfloor \frac{d^+(v)}{2} \rfloor$ and $\lceil \frac{d^+(v)}{2} \rceil$. ∎

Acyclic oriented graphs

Note that the stronger statement is crucial for the proof.



Part 1: On locally irregular oriented graphs

Part 2: We have $\chi'_{irr}(\overrightarrow{G}) \leq 6$ for every $\overrightarrow{G}$

Part 3: Deciding whether $\chi'_{irr}(\overrightarrow{G}) \leq 2$ is NP-complete

Part 4: Conclusion and open questions

A decomposition lemma

Lemma (B., Renault - 2013+)

Let $A_1 \cup A_2 \cup \ldots \cup A_k$ be a partition of $A(\overrightarrow{G})$. If

$$\chi'_{irr}(\overrightarrow{G}[A_1]) \leq x_1, \chi'_{irr}(\overrightarrow{G}[A_2]) \leq x_2, \ldots, \chi'_{irr}(\overrightarrow{G}[A_k]) \leq x_k,$$

then $\chi'_{irr}(\overrightarrow{G}) \leq \sum_{i=1}^{k} x_i$.

Proof. Let $\phi_1, \phi_2, \ldots, \phi_k$ be locally irregular x_1-, x_2-, ..., x_k-arc-colourings of $\overrightarrow{G}[A_1]$, $\overrightarrow{G}[A_2]$, ..., $\overrightarrow{G}[A_k]$, respectively. Then the arc-colouring ϕ of $\overrightarrow{G}$ defined as

$$\phi(\overrightarrow{a}) = (\phi_i(\overrightarrow{a}), i) \text{ for every } \overrightarrow{a} \in A(\overrightarrow{G}) \text{ such that } \overrightarrow{a} \in A_i,$$

induces $\sum_{i=1}^{k} x_i$ locally irregular subgraphs. ■

Decomposition into two acyclic oriented graphs

Lemma (B., Renault - 2013+)

There is a bipartition $A_1 \cup A_2$ of $A(\overrightarrow{G})$ inducing acyclic oriented graphs.

Proof. Denote $v_1, v_2, ..., v_n$ the vertices of $\overrightarrow{G}$. Then consider any arc $\overrightarrow{v_i v_j}$, and

$$\begin{cases} \text{add } \overrightarrow{v_i v_j} \text{ to } A_1 \text{ if } i < j, \\ \text{add } \overrightarrow{v_i v_j} \text{ to } A_2 \text{ otherwise.} \end{cases}$$

If $\overrightarrow{v_{i_1} v_{i_2} \ldots v_{i_k} v_{i_1}}$ were some circuit of $\overrightarrow{G}[A_1]$, then we would have $i_1 < i_k < i_1$. ■

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Since every acyclic oriented graph can be decomposed into three locally irregular subgraphs, we get the following via the previous two lemmas.

Corollary (B., Renault - 2013+)

We have $\chi'_{irr}(\overrightarrow{G}) \leq 6$ for every oriented graph $\overrightarrow{G}$.

Part 1: On locally irregular oriented graphs

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If our conjecture were true, would there be an easy classification of oriented graphs with irregular chromatic index at most 2?

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Theorem (B., Renault - 2013+)

Deciding whether $\chi'_{irr}(\overrightarrow{G}) \leq 2$ is NP-complete.

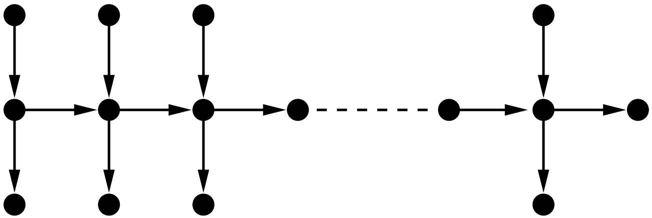
We show this by reduction from $\textsc{Not-All-Equal 3-Satisfiability}$.

Fiber gadget $\overrightarrow{F_2}$

We first introduce some "forcing" *fiber* gadgets $\overrightarrow{F_k}$.

Lemma (B., Renault - 2013+)

In every locally irregular 2-arc-colouring ϕ of $\overrightarrow{F_2}$, all of the even outputs of $\overrightarrow{F_2}$ have the same colour, while all of the odd outputs have the second colour. Besides, for every output $\overrightarrow{v_i' v_i}$ of $\overrightarrow{F_2}$, the vertex v_i' has outdegree 2 in the $\phi(\overrightarrow{v_i' v_i})$-subgraph.



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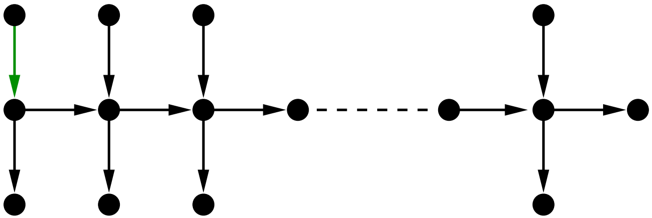

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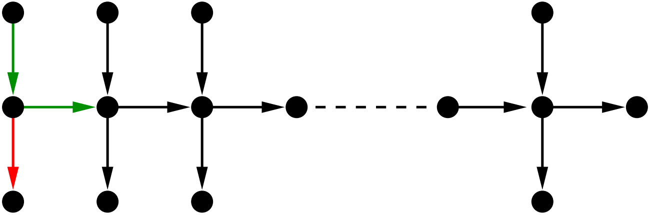

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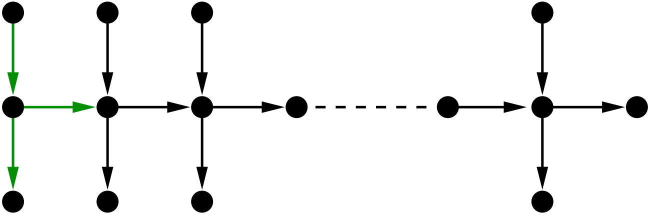

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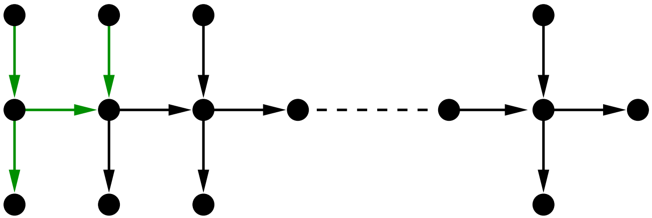

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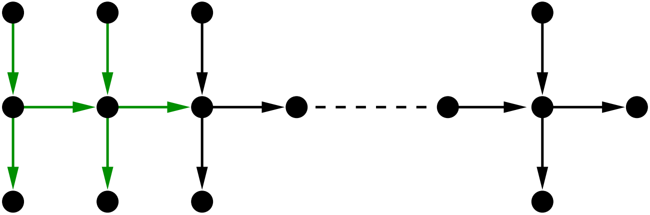

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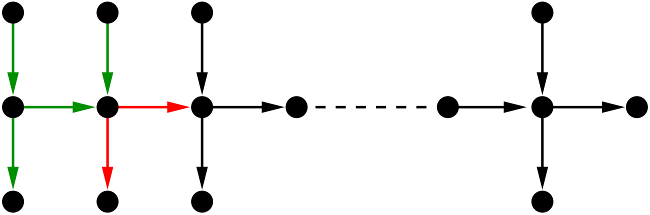

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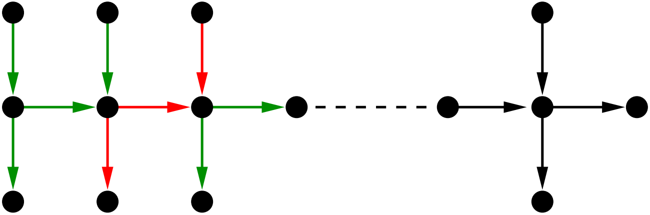

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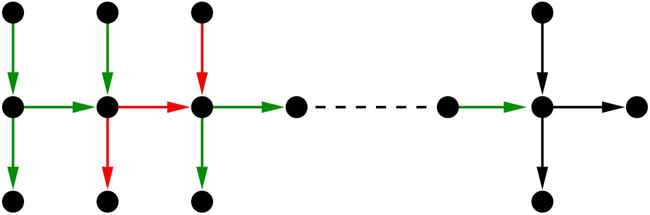

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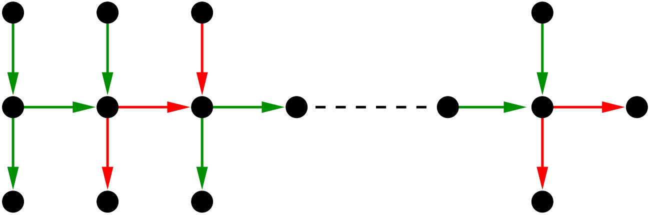

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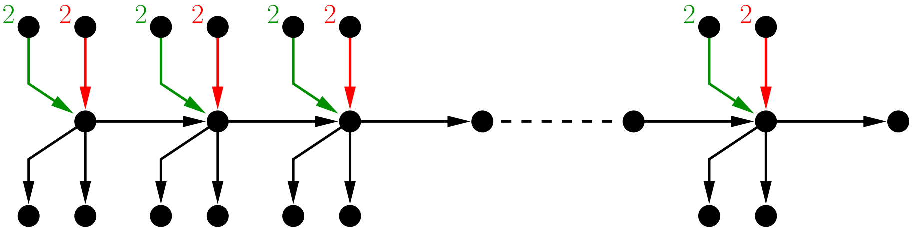

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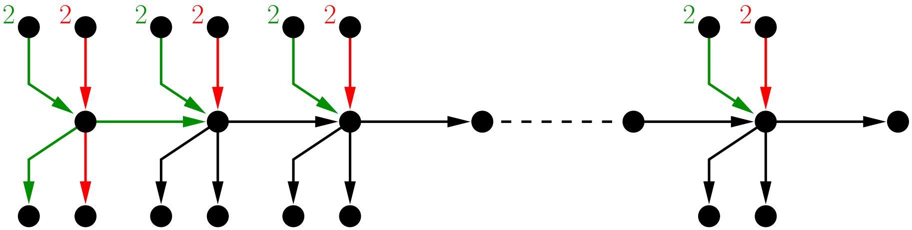

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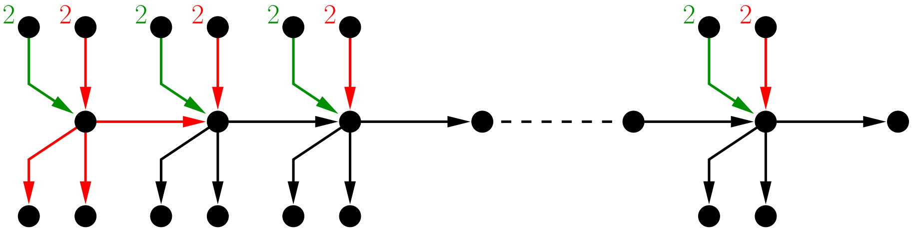

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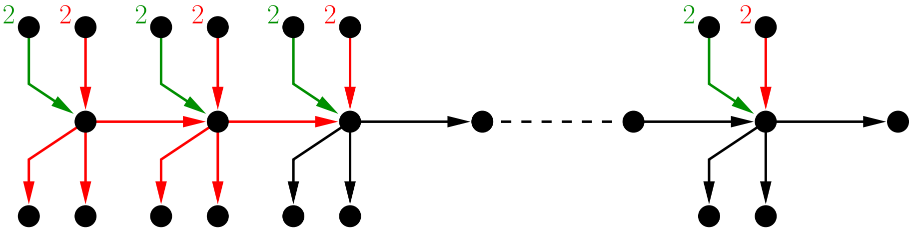

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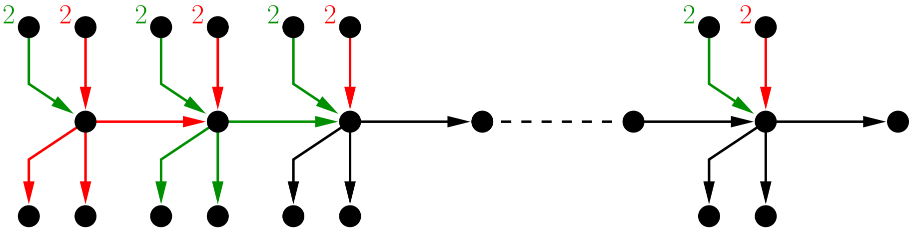

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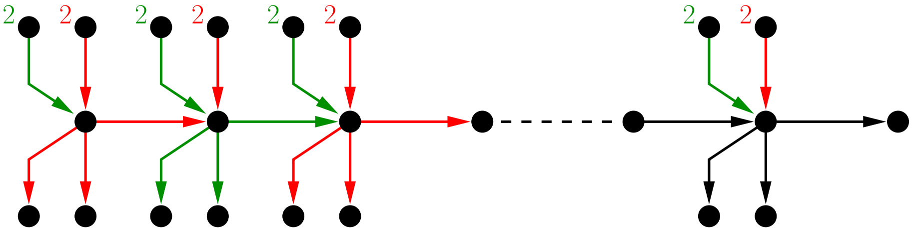

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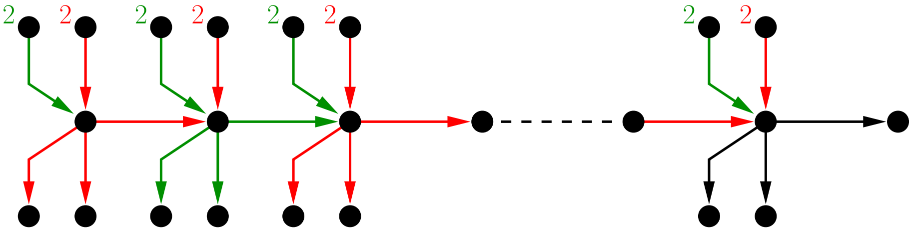

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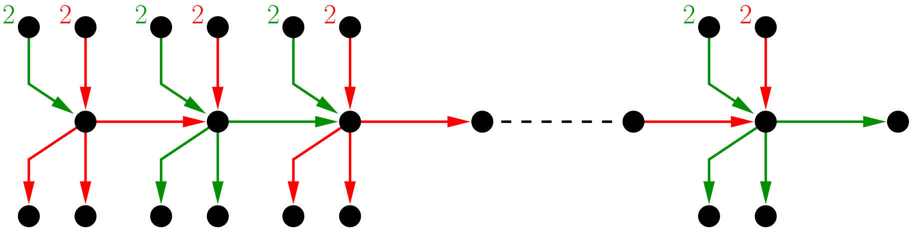

Deciding whether $\chi'_{irr}(\overrightarrow{G}) \leq 2$ is NP-complete

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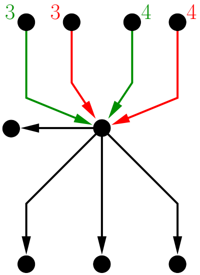

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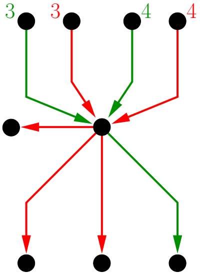

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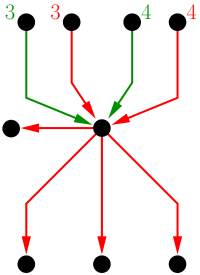

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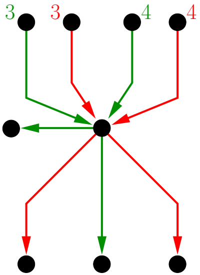

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Proof. For the clause gadget associated with C_j, we cannot have the three outputs coloured either all green or all red. This simulates the fact that all three variables of C_j cannot be all set to true or all set to false by a truth assignment of F.

The only thing we have to make sure of is that all arcs symbolizing the membership of a variable to the clauses which contain it have the same colour. This would simulate the fact that a variable provides the same truth value to every clause which contain it.

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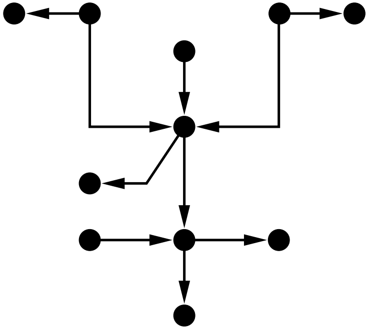
Proof. This is done using this gadget.



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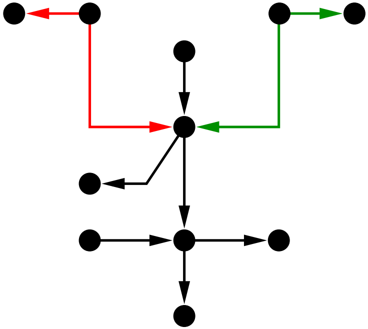

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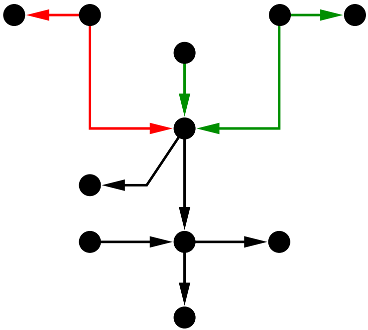

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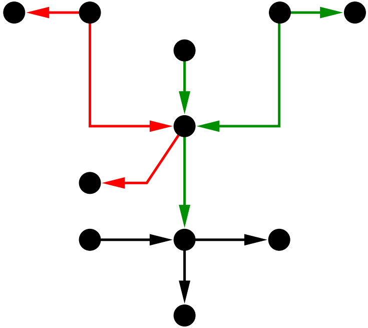

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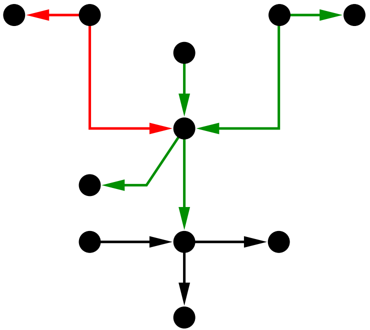

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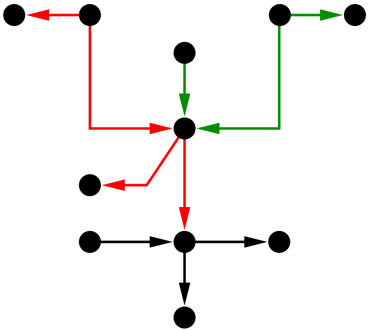

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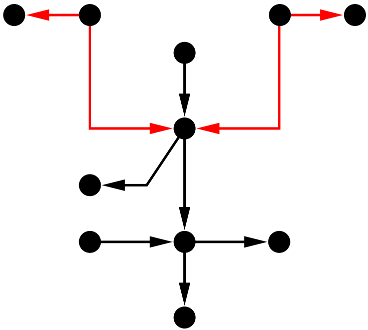

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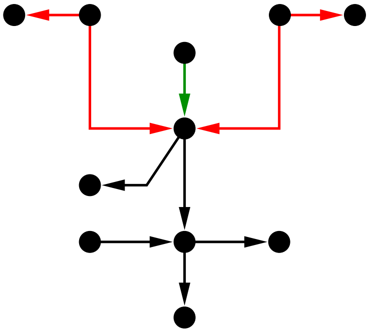

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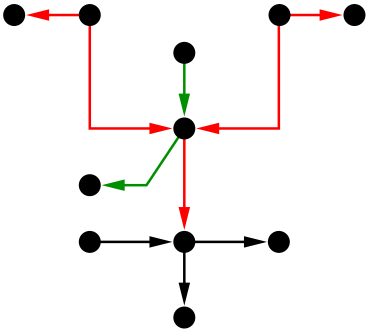

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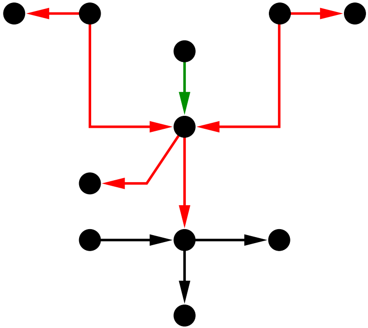

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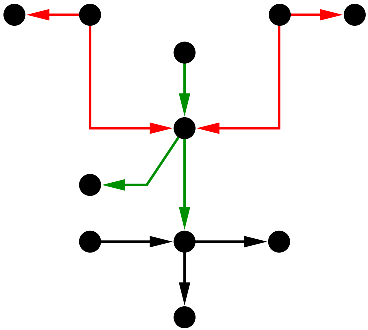

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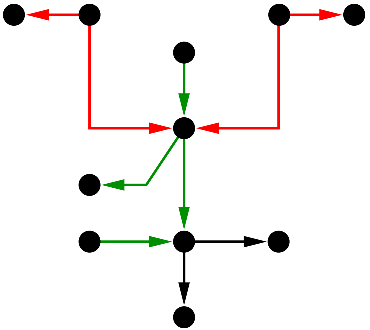

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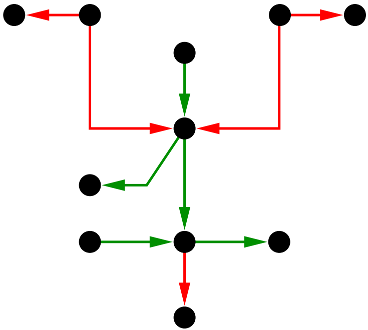

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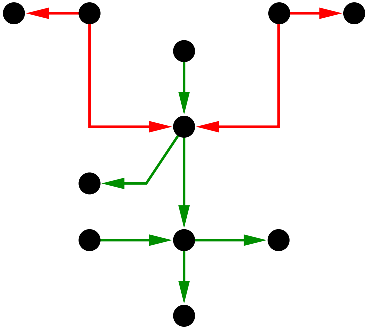

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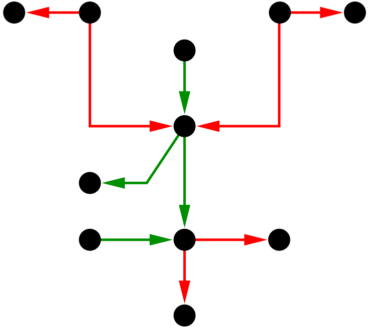

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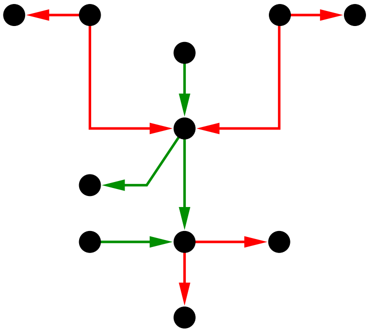

Part 1: On locally irregular oriented graphs

Part 2: We have $\chi'_{irr}(\overrightarrow{G}) \leq 6$ for every $\overrightarrow{G}$

Part 3: Deciding whether $\chi'_{irr}(\overrightarrow{G}) \leq 2$ is NP-complete

Part 4: Conclusion and open questions

Conclusion and open questions

- Our main conjecture still holds.
- What about tournaments?
- Analogous problems on oriented graphs?
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Thank you for your attention.