Strong edge-colouring of planar and bipartite graphs

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G: undirected simple graph
c: edge-colouring of G

Definition: *strong edge-colouring*

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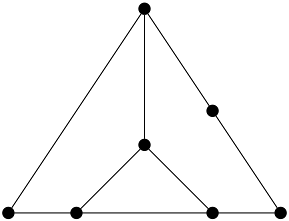

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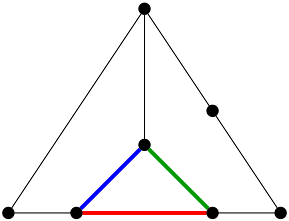

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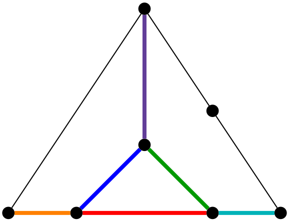

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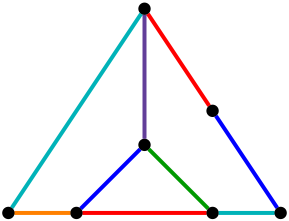

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Δ: maximum degree of an explicit graph

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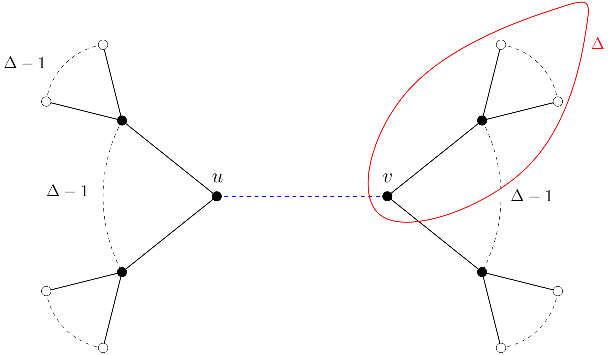
Brooks-like argument: $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ ($\approx 2\Delta^2$)



On the Brooks-like upper bound on χ'_s

optimality of $2\Delta^2$?

Conjecture [Erdős, Nešetřil – 1989]

We have $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 \text{ for } \Delta \text{ even, and} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \text{ otherwise.} \end{cases}$

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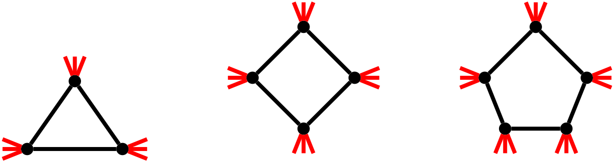
Theorem [Molloy, Reed – 1997]

If Δ is large enough, then $\chi'_s(G) \leq 1.998\Delta^2$.

Beyond Erdős and Nešetřil's conjecture

Less dependencies for graphs with no small cycles.



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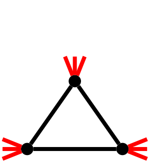

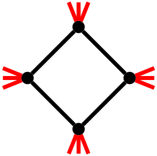

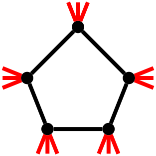

Theorem [Mahdian – 2000]

If G is C_4-free, then $\chi'_s(G) \leq (2 + o(1))\frac{\Delta^2}{\ln \Delta}$.

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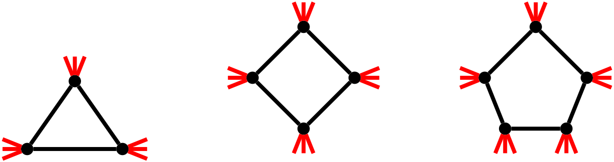

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Can we prove that for other graphs with no small cycles?

Planar graphs

g: minimum length of a cycle in an explicit graph

Theorem [Faudree, Gyárfás, Schelp, Tuza – 1990]

If G is planar, then $\chi'_s(G) \leq 4\Delta + 4$.

$4\Delta - 4$ should be the right upper bound.

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Improvable for sparse planar graphs, *i.e.* of large girth.

Theorem [B., Harutyunyan, Hocquard, Valicov – 2014]

	$\Delta \geq 7$	$\Delta \in \{5,6\}$	$\Delta = 4$	$\Delta = 3$
no girth restriction	4Δ	$4\Delta + 4$	$4\Delta + 4$	$3\Delta + 1$
$g \geq 4$	4Δ	4Δ	$4\Delta + 4$	$3\Delta + 1$
$g \geq 5$	4Δ	4Δ	4Δ	$3\Delta + 1$
$g \geq 6$	$\mathbf{3\Delta + 1}$	$\mathbf{3\Delta + 1}$	$\mathbf{3\Delta + 1}$	3Δ
$g \geq 7$	3Δ	3Δ	3Δ	3Δ

Bipartite graphs

Conjecture [Faudree, Gyárfás, Schelp, Tuza – 1990]

If G is bipartite, then $\chi'_s(G) \leq \Delta^2$.

Reached e.g. for any complete bipartite graph $K_{a,a}$.

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$G = (A, B, E)$: bipartite graph with bipartition A and B
(Δ_A, Δ_B)-bipartite graph: A and B have maximum degree Δ_A and Δ_B, resp.

Conjecture [Brualdi, Quinn Massey – 1993]

If G is (Δ_A, Δ_B)-bipartite, then $\chi'_s(G) \leq \Delta_A \Delta_B$.

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We confirm the first conjecture for $\Delta_A = 3$ and $\Delta_B \geq 4$.

Theorem [B., Lagoutte, Valicov – 2014+]

If G is $(3, \Delta_B)$-bipartite, then $\chi'_s(G) \leq 4\Delta_B$.

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- ... but $3\Delta_B$ should be the right upper bound.

- The conjecture is still open for general bipartite graphs...
- ... can our proof scheme be generalized?

Thank you for your attention.