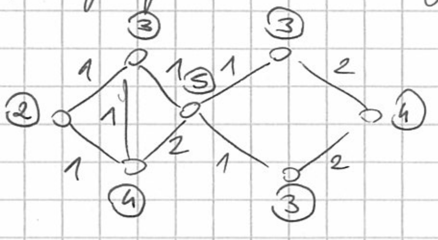
1-2-3 Conjecture & Variations

I) 1-2-3 Conjecture + motivation –

G simple graph w/ k-edge-weighting w



$\Rightarrow$ compute "incident sums"
$\Rightarrow$ vertex-colouring of σ. I

if w yields σ proper $\rightarrow$ w sum-colouring.
least k s.t. G has a sum-colouring k-edge-weighting (if any) denoted $\chi^e_{\Sigma}(G)$.

A) Facts.

$\chi^e_{\Sigma}(G)$ undefined if and only if G has isolated K_2's.
G nice: no such component.
fact: every nice graph is sum-colouring edge-weightable.

How many weights for nice graphs?

1-2-3 Conjecture – Karoński, Łuczak, Thomason (2004)
For every nice graph G, have $\chi^e_{\Sigma}(G) \leq 3$.

Best possible: e.g. $\chi^e_{\Sigma}(C_{4k+2}) = 3$.

More generally:

Theorem – Dudek, Wajc (2011).
Deciding whether $\chi^e_{\Sigma}(G) \leq 2$ is NPC.
$\hookrightarrow$ even for cubic

But ....

Theorem – Addario-Berry, Dalal, Reed (2008)
For $G \in G_{n,p}$, a.a.s. $\chi^e_{\Sigma}(G) \leq 2$.

$\uparrow$
any p works.

(B) Bounds on χ^e_{Σ}

Successive bounds on χ^e_{Σ} ... all constants.

183	30	16	13	5
Karoński, Łuczak, Thomason (2004)	Addario-Berry, Dalal, McDiarmid, Reed, Thomason (2007)	Addario-Berry, Dalal, Reed (2008)	Wang, Yu (2008)	Kalkowski, Karoński, Pfender (2010)
probabilistic	decompositions	decompositions	decompositions	algorithmic.

$\chi^e_{\Sigma} \leq 5$: proof derived from total version.

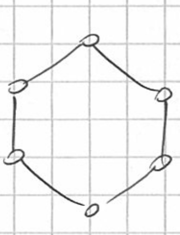
(C) Verification of the 1-2-3 Conjecture

true for :
- bipartite
- 3-colourable
- some planar
- complete
- cartesian products (some)
- ...

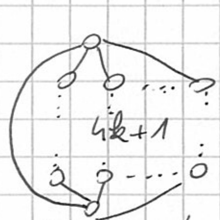
for bipartite : characterization of those where 1,2 suffice?

$\Rightarrow$ not clear.

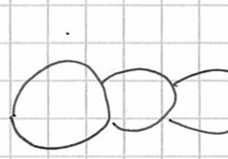
- NPC of the pbm for bipartite would mean "no".
- on the other hand, only a few classes needed 1,2,3 are known :



C_{4k+2}



θ-graphs



polygon~~al~~-trees

My guess : $\chi^e_{\Sigma}(\text{bip}) \leq 2$ iff obtained from C_{4k+2} by adding ears of length $4k+1$ joining ~~adjacent vertices w/ same degree~~ some vertices verifying some conditions. (same degree + adj?)

Idea : $P_{4k+1} \sim$ edge when 1,2 are used.

Support: - 3-connected bipartite $\Rightarrow \chi^e_{\Sigma} \leq 2$ Lu, Yu, Zhang (2010)

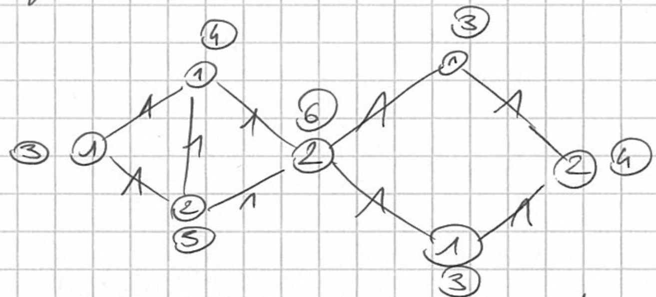
(+) if one even part $\Rightarrow \chi^e_{\Sigma} \leq 2$ Chang, Lu, Wu, Yu (2011)
$\Rightarrow$ true for trees

So only odd, odd are to be considered.

Q: What for connectivity-1 graphs?

(II) Total version : 1-2 Conjecture

Also weight the vertices $\Rightarrow$ count every vertex' weight in its sum



same terminology ... || sum-colouring total-weighting $\chi^t_{\Sigma}(G)$

Now $\chi^t_{\Sigma}(G)$ defined for every G.

Note: $\chi^t_{\Sigma}(G) \leq \chi^e_{\Sigma}(G)$ and weighting the vertices can only help...

1-2 Conjecture - Przybyło, Woźniak (2010)
For every graph G, have $\chi^t_{\Sigma}(G) \leq 2$

Verified for many classes of graphs .. bipartite $\rightarrow$ easy.
Here a better bound ...

Theorem - Kalkowski (2015)
For every graph G, have $\chi^t_{\Sigma}(G) \leq 3$

Proof: use 1,2,3 on edges, and 1,2 on vertices
- start from 2 on edges, 1 on vertices.
- process the vertices linearly, defining two possible sums a and f where $f = a+1$
- at any moment of the process, every sum is a or f.
- when treating a vertex, modify backward weights w/ preserving

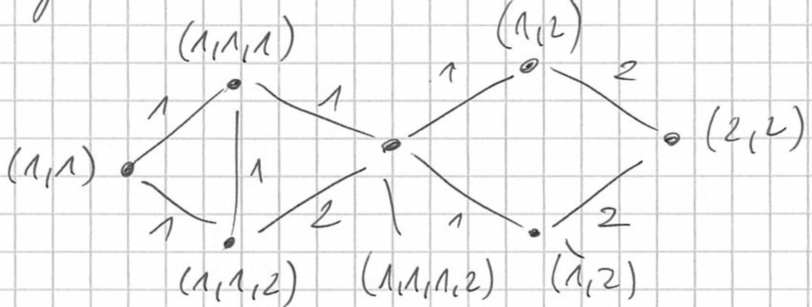
- at the end, do $+1$ on every vertex where $\sigma = a$

Proving $\chi^e_{\Sigma} \leq 5$, pretty much the same:

- cannot modify at the end
- two values for every vertex, $\neq$ from the neighbours'
- play on edges w/ staying in "safe" ones.

III Multiset version.

Would give:



... multiset-colouring and χ^e_M. nice graphs ...

Conjecture - Addario-Berry, Dalal, Reed (2005)
For every nice graph G, have $\chi^e_M(G) \leq 3$

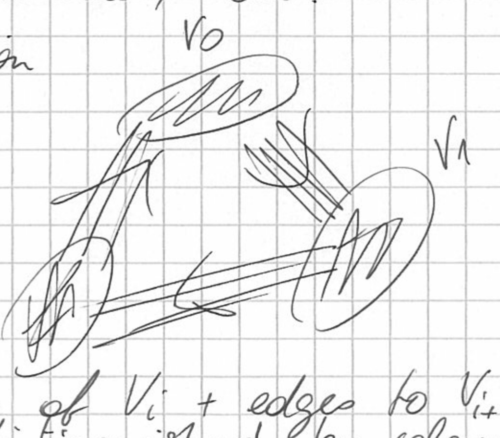
Note: have $\chi^e_M(G) \leq \chi^e_{\Sigma}(G) \leq 5$. ← Many results apply.

But ... (II) does not.

Theorem - Addario-Berry, Dalal, Reed (2005)
Have $\chi^e_M(G) \leq 4$.

Proof: if G 3-colourable, ok.

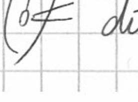
otherwise a partition ← unfriendly partition

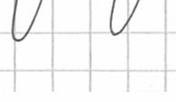


w/
(1) $\forall v \in V_i$, more neighbs in V_{i+1} than V_i.
(2) $\forall v \in V_i$, edge to V_{i+1} ✓2

first fit.

Colour i all edges of V_i + edges to V_{i+1} so that neighbours of V_i distinguished by colour i.
Put all others colour 4.


dist by i


dist by $i+1$.

+ complexity & things for regular

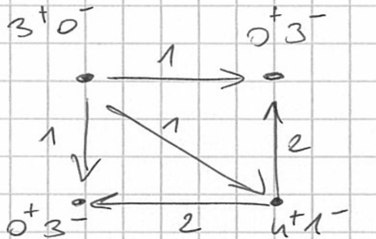
Again the same question for bipartite ...
Here:

Theorem – Havet, Paramaguru, Sampathkumar (2014)

For every nice bipartite graph G w/ $\delta(G) \geq 3$,
have $\chi^e_\mu(G) \leq 2$

(IV) Directed versions.

Digraph D w/ arc-weighting



two kinds of sums: outgoing (σ^+) and ingoing (σ^-)

if do $\sigma^+ + \sigma^-$ at every vertex and ask for distinction $\Rightarrow$ 1-2-3 Conjecture.

"Good" variant regarding both the 1-2-3 Conjecture and the directed context?

① Relative sums $\sigma^+ - \sigma^-$ Borowiecki, Grytczuk, Pilśniak (2012)

1,2 suffice even in lists.

② Single-type sums σ^+ Baudon, B, Sopena (2015)

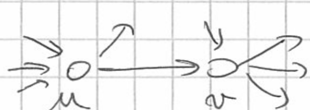
1,2,3 ok and tight

Very easy: induction

$d^+ \geq d^-$

equivalent of 1,2 Conjecture not true.

③ "Łuczak sums" Barme, B., Przybyło, Woźniak (2015+)



$\sigma^+(u) \neq \sigma^-(v)$

× makes sense.
× exceptions.
× proof $\rightarrow$ equivalent to 1-2-3 Conjecture in bipartite graphs

(V) And a lot more ...

- proper version (neighbor-sum-dist. index)
- list versions. (w/ combinatorial Nullstellensatz)
- product version. ← coprime numbers ~ multiset + neutral.
- locally irregular decompositions.
  ↳ same as multiset/sum in regular graphs w/ two colours
- mix everything ...

χ^{e}_{Π} 4