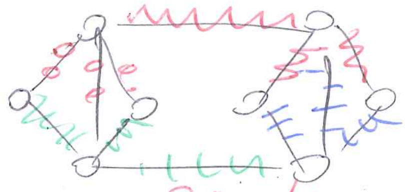
Decomposing highly edge-connected graphs into paths

(1

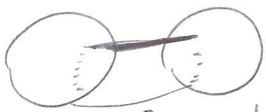
Decomposition : partition of edges



Barát / Thomassen

For every tree T, there is c_T st for every c_T-edge-co graph G w/ $\|G\|$ divisible by $\|T\|$, $\exists$ T-decompo.

Not true if T = cycle



Verified for | stars / some histars / P_3 / P_4 / P_{2^k} → Thomassen
trees w/ diam ≤ 4 → Merker

→ plus general :
copies homomorphes de T
⇒ if diam$(T) \leq$ girth(G)
Donc ok pour graphes de large girth

P_5 Botler Mota Oshiro Wakabayashi

P_ℓ "

But Complicated...
Easier proof → Actually δ more important for paths...

Th. G 24-edge-co, $\ell \geq 2$
large degree ⇏ P_ℓ-decompo.

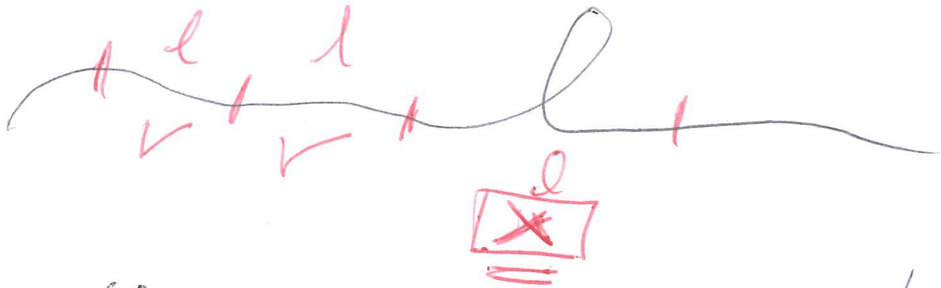
Best possible? ⇒ $\boxed{3}$

2 not possible according to construction...

Proof idea:

Could theoretically work for 8 .....

Pick a eulerian tour, and decompose along it

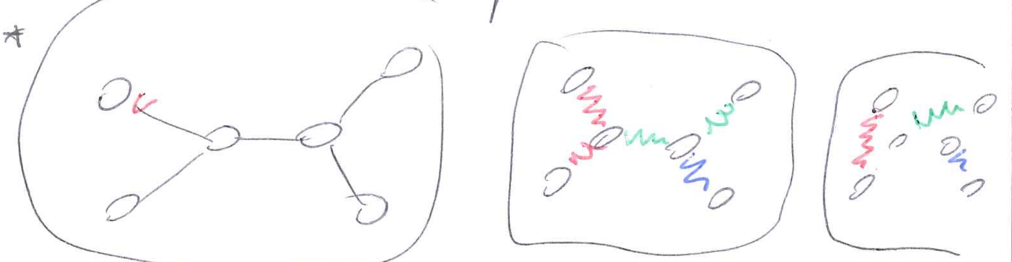


Solution (kinda'): pre-decompose G into paths...

⇒ main notion path-graph H on G.

$H = (V, \mathcal{P})$ where $\mathcal{P}$ partitions E into paths.

From H, H^*



G H

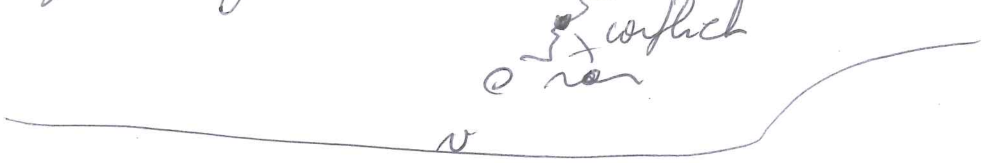
term:

- $d_H(v) = d_{H^*}(v)$
- H connected/tree ⇒ H^* connected/tree
- H eulerian ⇒ H connected, all degrees even.
- H tour ⇒ tour in H^*.

If H path-graph, the patching procedure above has more chances to succeed — except that consecutive paths may have common vertices...

and no control on lengths so far.

But... if all lengths are $\geq \ell$ ($(\geq \ell)PG$), conflicts get very local, i.e. around the vertices

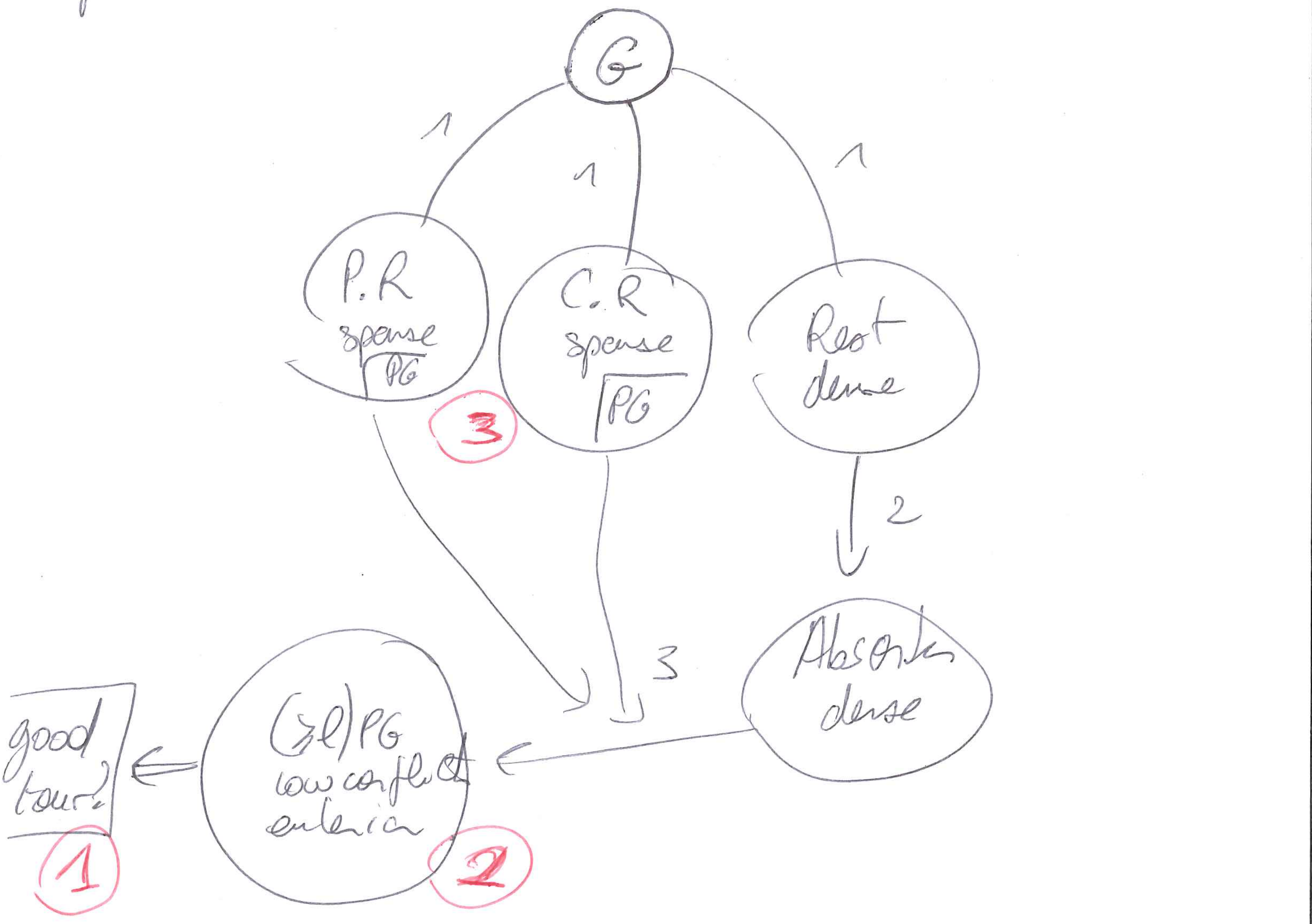


and if low conflict, hopefully a conflictless eulerian tour.

Back to G. We can express, assuming δ large, G as an $\Big\|$ $(\geq \ell)PG$, very low conflict ... But Eulerian???

__Solution__: build an __absorber__ w/ these properties that is so strong that, if we make it eulerian, cannot spoil the conflicts too much.

If not eulerian, because of __connectedness__ a/ad __degree__



(A) low conflicts eulerian $\Rightarrow$ conflictless eulerian tour.

notion of conflicts ... ?

conf ration$_v$ (P) max conflict pourcentage.

conf ration (H) ... max.

Result for conf $\leq \frac{1}{8}$ (But Jackson proved $\frac{1}{2}$).

<u>Proof</u>: first pair the paths around the vertices to get a set of "safe" transition.

$\Rightarrow$ for that, conflict graph dense $\Rightarrow$ Ham. cycle in the complement.

Start from some vertex and build conflictless tours. Say t of them. $\underline{t=1} \Rightarrow$ good.

then $t \geq 2$. $\Rightarrow$ modify the pairing around a vertex traversed by 2 tours.

Possible because conf $\leq \frac{1}{8}$. $\square$

Ⓑ Result we use

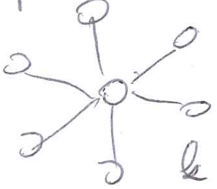
$$H\begin{cases}\bullet\ q\text{-path-graph}\\ \bullet\ \alpha\text{-dense}\\ \bullet\ \text{conflict } c.\end{cases} \Rightarrow H'\begin{cases}\bullet\ 2q\text{-path-graph}\\ \bullet\ \frac{\alpha}{5}\text{ dense}\\ \bullet\ \text{conflict } 16cq\end{cases}$$

constant

<u>Proof</u>: Random pairing process...
LLL + Chernoff $\square$

In particular

<u>graph G</u> $\begin{cases}\bullet\ 1\text{-PG}\\ \bullet\ 1\text{-dense}\\ \bullet\ \text{mult } \frac{1}{\delta}\end{cases} \Rightarrow \begin{cases}\bullet\ 2^p\text{-PG}.\\ \bullet\ \frac{1}{5^p}\text{ dense}\\ \bullet\ C \text{ under any constant provided } \delta \text{ large enough.}\end{cases}$



Ⓒ <u>Context</u>:

$H\begin{cases}(\geq \ell)\text{ PG}\\ \text{low conflicts}\end{cases} \Rightarrow$ make it ① Connected ② Even. without spoiling conflicts

① Just add a <u>sparse</u> for conflict $(\geq \ell)$ path tree (= PG being a tree)

② How to repair ~~conflicts~~ degree parities? pair them and add a joining path

In a tree, possible to find a system of edge-disjoint paths joining pairs of marked vertices...

So a $(\geq \ell)$ PT... but what to do w/ things not added?

$\Rightarrow$ decomposable. Possible if all paths are multiple of ℓ $\Leftarrow$ $(\ell, 2\ell)$ tree
$\longrightarrow$ + bounded degree

So obtain $(\ell, 2\ell)$ trees w/ bounded degree?

1. 2 edge co $\Rightarrow$ $(1,2)$ tree subcubic.
2. $(1,k)$ tree + bounded Δ + source of degree $\Rightarrow$ $(1,k+1)$ w/ bound Δ
3. $(1,k+1)$ tree + bounded Δ + degree $\Rightarrow$ $(\ell, 2\ell)$ tree bounded Δ

proof by induction. Extend paths using source of degrees.

Final picture. Assume ℓ even.

G 8edge co
large degree

⇓ N.W.

$\vec{G}$ orientation 4 arcstrong
balanced

⇓ Edmonds

T_1 T_2 T_3 T_4
spanning trees
of G $\frac{1}{2}$ sparse

\+

Rest
$\frac{1}{2}$ dense

pair $T_1 + T_2$

2edge co
$\frac{1}{2}$ Sparse

pair $T_3 + T_4$

2edge co
$\frac{1}{2}$ Sparse

sparse degree feeding

Sparse
CR

Sparse
PR

$(\leq \ell)$PG
dense
low conflicts

conclude!

More words

- Same 2 if not using (PR) $\Rightarrow$ ok if eulerian from the beginning. So || k edge co / Eulerian / large degree $\Rightarrow$ eulerian hom w/ large girth

- Same CCL for Eulerian + large degree ...

- 3-edge-co also

How to go to decompo arguments