Imposing vertex-part restrictions while arbitrarily partitioning a graph

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A simple problem

We want to share a network of connected resources between an arbitrary number of users. For performance sake, we want to do it in such a way that the following two conditions are fulfilled:

- → A resource must be attributed to only one user.
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Thus, networks which are interesting for us are those which can be partitioned into an arbitrary number of arbitrary size in such a way that the above two statements are respected.

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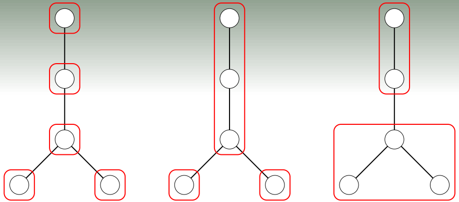
According to this definition, networks which interest us the most in regard to our problem are those whose topology forms an *arbitrarily partitionable graph*:

Definition

The graph G is said to be *arbitrarily partitionable* (AP for short) if every partition of n is realizable in it.

Examples

Some realizations in the graph $P(1,1,2)$:



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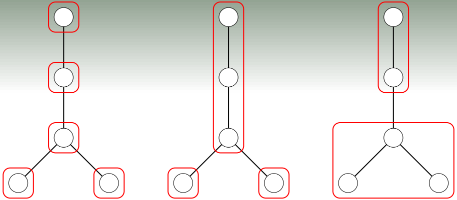
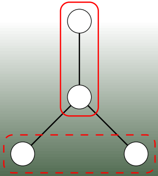


The case of the graph $P(1,1,1)$:



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→ Every *traceable* graph is AP.

. . . but is tough to determine!

→ The complexity of determining whether a graph is AP is still unknown.

→ Deciding if a partition is realizable in a graph is NPC [Rob98].

→ An integer n admits a number of partitions exponential in n [FS09].

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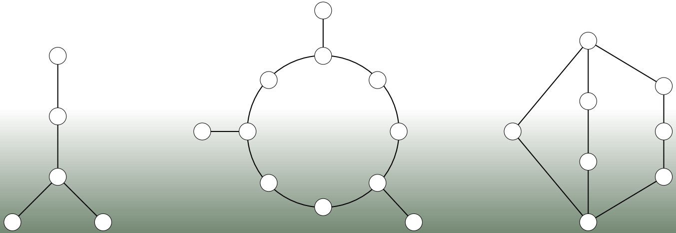

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An important result which will be needed later on:

Theorem [Barth and Fournier, 2006 [BF06]]

An AP tree has maximum degree at most four. Moreover, any degree 4 node in such a tree is necessarily adjacent to a leaf.

What is up?

The idea comes from the following theorem which ensures that any k-connected graph can always be partitioned in k connected parts, even if we impose k *vertex-part restrictions*:

Theorem [Lovàsz, 1977 [Lov77] - Györi, 1978 [Gyo78]]

Let G be a k-connected graph with order n, $\tau = (\tau_1, \dots, \tau_k)$ a partition of n, and $v_1, \dots, v_k$ distinct vertices of G. There necessarily exists a realization $V_1, ..., V_k$ of τ in G such that $v_i \in V_i$ for every $i \in [1, k]$.

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Question: does there exist some AP graphs in which every partition can be realized if we are given an arbitrary number of vertex-part restrictions?

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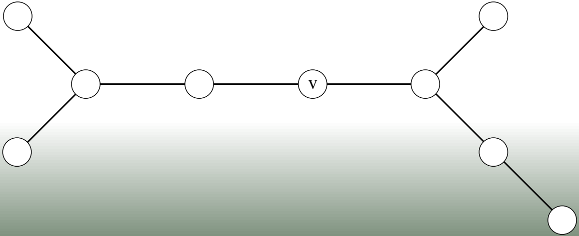

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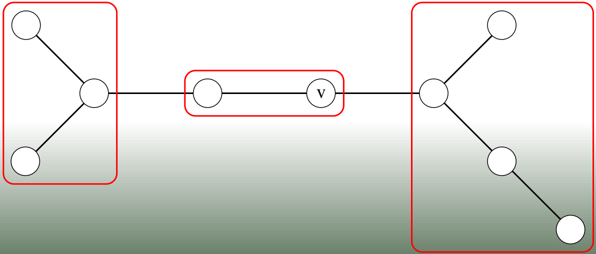

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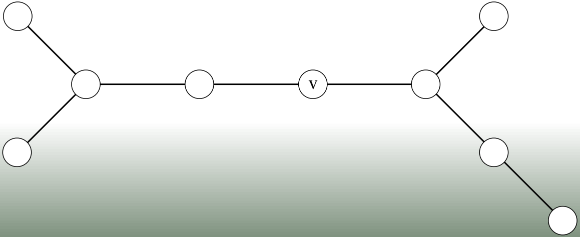

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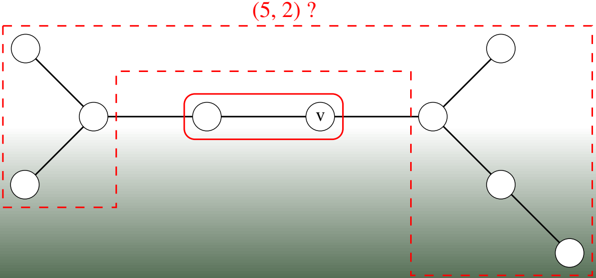

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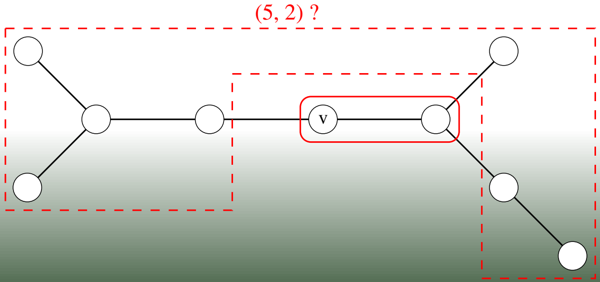

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Definition

The graph G is said to be *arbitrarily partitionable under one restriction* (wAP+1 for short) if every partition τ of n is realizable in it under every restriction of $V \times \tau$.

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One has to see that, for any restriction (v, τ_i), the part of the realization containing v can be different for two sequences τ and τ'. It would be easier to always pick the same one...

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- → $v \in S_q$,
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The graph G is said to be *strongly arbitrarily partitionable under one restriction* (sAP+1 for short) if every vertex $v \in V$ is q-imposable in G for every $q \in [1, n]$.

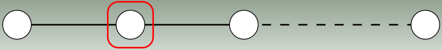
sAP+1 $\neq$ wAP+1

Note that there exists some graphs which are wAP+1 but not sAP+1.

Example: the 4-balloon $B(1, 1, 4, 6)$.

First observations

A chain is not sAP+1 if it is with order more than three:



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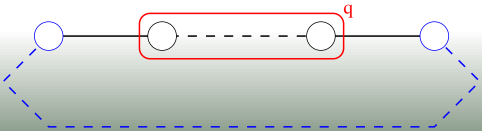
However, cycles do not have this problem:



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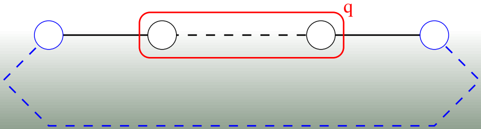

Thus, any hamiltonian graph is sAP+1!

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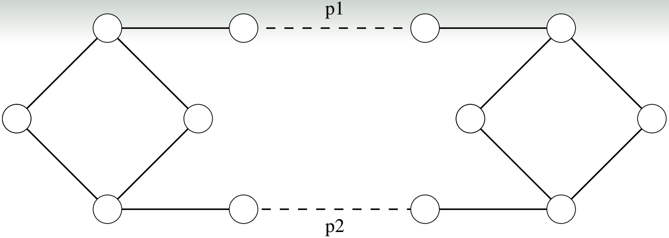

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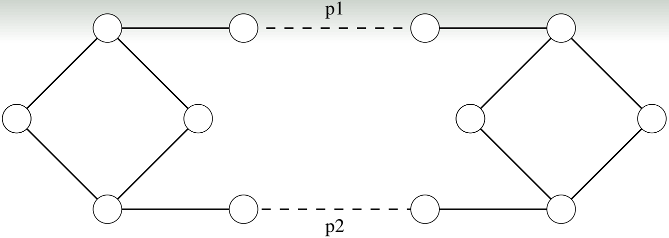

The cylinder $C(p_1, p_2)$

Theorem

Let $p \geq 1$ be an integer. The cylinder $C(p, p)$ is sAP+1 iff p is even.

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Properties

If the 4-balloon $B(b_1, b_2, b_3, b_4)$ if sAP+1, then:

- → $b_1 = 1$,
- → $P(1, b_2, b_3, b_4)$ is AP,
- → $P(b_2, b_3, b_4)$ is AP,
- → $\mathrm{pgcd}(b_2 + 1, b_3 + b_4 + 1) = 1$,
- → at least two values of the set $\{b_2, b_3, b_4\}$ are even,
- → at least one value of the set $\{b_2, b_3, b_4\}$ is a multiple of 4.

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Can we find general conditions for a graph to be wAP+1? sAP+1?

And if we impose more vertex-part restrictions?

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Thanks for your attention!

Questions?