On the size of graphs that can be partitioned under a given number of prescriptions

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4th Polish Combinatorial Conference
September 20th, 2012

Part 1: Partitioning graphs under prescriptions (AP+k graphs)

Part 2: Results on powers of traceable or Hamiltonian graphs

Part 3: On the minimum size of an AP+k graph

Part 4: Conclusions

Our problem

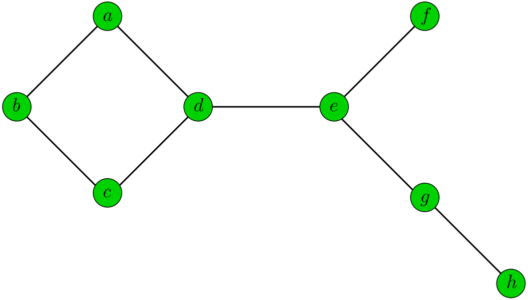
We want to share a network of distinct resources to an arbitrary number of users in such a way that the following requirements are met.

- A resource is attributed to exactly one user.
- Resources of a same subnetwork must be able to communicate within it.
- Some users are each allowed to request a specific resource.

Example

As an example, let us consider the following resource demand and network.

User 1: (1, e) User 2: 2 User 3: 2 User 4: 3



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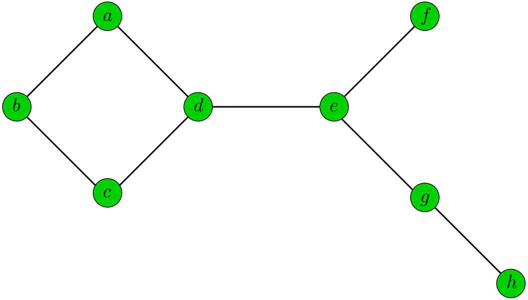

Can we satisfy our users?

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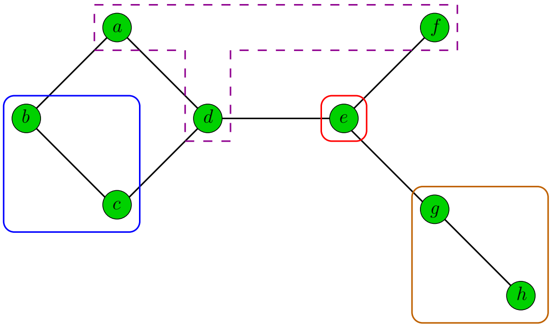

No! We cannot meet all our constraints under this vertex membership constraint.

Formalizing using graph theory

Let G be a connected graph on n vertices.

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Definition: *realizable sequence*, *realization*

A sequence $\tau = (n_1, ..., n_p)$ adding up to n is *realizable in* G if there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G on n_i vertices. The partition $(V_1, ..., V_p)$ is called a *realization of* τ *in* G.

In the previous talk, Olivier told you about graphs that can be partitioned in this way following every sequence summing up to their respective order.

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We here strengthen our definitions with a new membership constraint.

Definition: *k-prescription*, *realization under prescription*

A k-tuple $(v_1, ..., v_k)$ of pairwise distinct vertices of G is called a *k-prescription of* G. If $p \geq k$ and there exists a realization $(V_1, ..., V_p)$ of τ in G such that for every $i \in [1, k]$ we have $v_i \in V_i$, then τ is *realizable in* G *under* $(v_1, ..., v_k)$.

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Notice that we consider that the k part sizes associated to our k prescribed vertices are the first k of the sequence.

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We finally introduce our main definition.

Definition: *AP+k graph*

If every sequence adding up to n consisting of more than k elements is realizable in G under every k-prescription, then G is *arbitrarily partitionable under k-prescriptions*.

In our introducing problem, allowing k special users to request a resource is only possible if our network has an AP+k graph topology.

Formalizing using graph theory

Definition: *AP+k graph*

If every sequence adding up to n consisting of more than k elements is realizable in G under every k-prescription, then G is *arbitrarily partitionable under k-prescriptions*.

In other words, an AP+k graph is partitionable into two different kinds of parts.

- Exactly k *prescribed parts*, which must fulfil the vertex membership, connectivity and size constraints.
- Maybe some additional *free parts*, which must only satisfy the connectivity and size constraints.

Prescription and connectivity

Notice that a graph must be connected enough to be AP+k.

Observation

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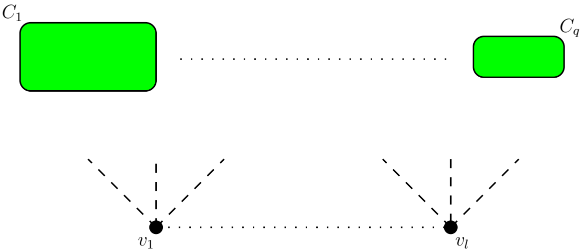

Let us consider that $l \leq k$.

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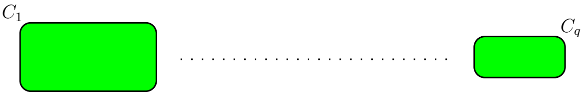

This subgraph is not connected and cannot be partitioned following $(\sum_{i=1}^{q} |C_i|)$.

Hence, we cannot realize $(1, ..., 1, \sum_{i=1}^{q} |C_i|)$ in this graph under $(v_1, ..., v_l)$. If $l < k$, then one can prescribe some extra vertices to parts with size 1 until the prescription has size k.

Part 1: Partitioning graphs under prescriptions (AP+k graphs)

Part 2: Results on powers of traceable or Hamiltonian graphs

Part 3: On the minimum size of an AP+k graph

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Partitioning powers of traceable or Hamiltonian graphs

We proved the following two results.

Theorem 1 (Baudon, B., Przybyło, Woźniak, 2012)

The graph P_n^k is AP+$(k-1)$ for every $k \geq 1$ and $n \geq k$.

Theorem 2 (Baudon, B., Przybyło, Woźniak, 2012)

The graph C_n^k is AP+$(2k-1)$ for every $k \geq 1$ and $n \geq 2k$.

These results are sharp regarding the connectivity of the corresponding graphs.

Theorem 1 (Baudon, B., Przybyło, Woźniak, 2012)

The graph P_n^k is AP+$(k-1)$ for every $k \geq 1$ and $n \geq k$.

Is there a realization of $(n_1, ..., n_p)$ in P_n^k under $(v_{i_1}, ..., v_{i_{k-1}})$?

This result is proved by induction on k. For a given value of k, we construct a part V_1 in such a way that:

- it has size n_1, induces a connected subgraph of P_n^k, and contains v_{i_1};
- the remaining graph $P_n^k - V_1$ is the $(k-1)^{st}$ power of a path.

By the induction hypothesis, we may next find a realization $(V_2, ..., V_p)$ of $(n_2, ..., n_p)$ in $P_n^k - V_1$ under $(v_{i_2}, ..., v_{i_{k-1}})$. It follows that $(V_1, ..., V_p)$ is a correct realization of $(n_1, ..., n_p)$ in P_n^k under $(v_{i_1}, ..., v_{i_{k-1}})$.

Partitioning powers of cycles

Theorem 2 (Baudon, B., Przybyło, Woźniak, 2012)

The graph C_n^k is AP+$(2k-1)$ for every $k \geq 1$ and $n \geq 2k$.

Is there a realization of $(n_1, \ldots, n_p)$ in C_n^k under $(v_{i_1}, \ldots, v_{i_{2k-1}})$?

Notice that consecutive vertices of C_n^k taken along its underlying cycle induce the k^{th} power of a path. Hence, the idea here is to divide our graph into k^{th} powers of paths in such a way that previous Theorem 1 can be used.

More precisely:

- they should not contain too many prescribed vertices;
- they must have enough vertices so that the prescribed parts associated to the prescribed vertices they contain can be picked from them.

In some cases, additional prescriptions may also be used so that the union of parts located into adjacent paths induces a single connected part.

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A lower bound on the size of an AP+k graph

Recall that an AP+k graph must be $(k+1)$-connected.
Hence, we deduce the following.

Observation

If G is an AP+k graph on n vertices, then $\|G\| \geq \lceil \frac{n(k+1)}{2} \rceil$.

An AP+k graph whose size meets this lower bound is an *optimal AP+k graph*.

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We here only focus on the **existence** of optimal AP+k graphs on n vertices for every $k \geq 1$ and $n \geq k$.

Harary graphs

Harary provided a construction which yields a k-connected graph with order n whose size is $\lceil \frac{nk}{2} \rceil$ for arbitrary k and n.

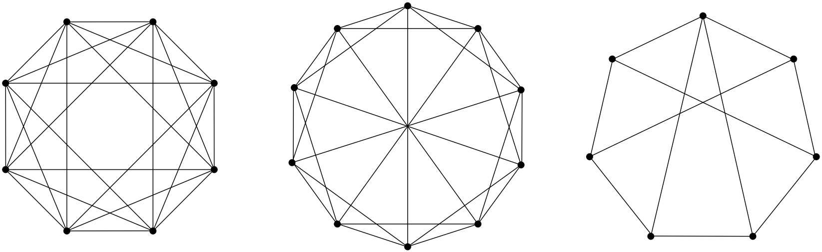
Definition: *Harary graph*

Let $k \geq 1$ and $n \geq k$ be two integers. The k-connected Harary graph on n vertices, denoted by $H_{k,n}$, has vertex set $\{v_0, ..., v_{n-1}\}$ and the following edges:

- if $k = 2r$ is even, then two vertices v_i and v_j are linked if $i - r \leq j \leq i + r$;
- if $k = 2r + 1$ is odd and n is even, then $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in [0, \frac{n}{2} - 1]$;
- if $k = 2r + 1$ and n are odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$;

where the subscripts are taken modulo n.

Some examples of Harary graphs



The Harary graphs $H_{6,8}$, $H_{5,10}$, and $H_{3,7}$

Partitioning Harary graphs under prescriptions

Observe that $H_{k,n}$ is isomorphic to $C_n^{k/2}$ for every even $k \geq 2$ and $n \geq k$. Hence, the following holds.

Corollary of Theorem 2

The Harary graph $H_{k,n}$ is AP+$(k-1)$ for every even $k \geq 2$ and $n \geq k$.

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The Harary graph $H_{k,n}$ is AP+$(k-1)$ for every even $k \geq 2$ and $n \geq k$.

We proved the following.

Theorem 3 (Baudon, B., Sopena, 2012)

The Harary graph $H_{2k+1,n}$ is AP+$2k$ for every $k \neq 1$ and $n \geq 2k+1$.

Partitioning Harary graphs under prescriptions

Theorem 3 (Baudon, B., Sopena, 2012)

The Harary graph $H_{2k+1,n}$ is AP+$2k$ for every $k \neq 1$ and $n \geq 2k+1$.

Is there a realization of $(n_1, \ldots, n_p)$ in $H_{2k+1,n}$ under $(v_{i_1}, \ldots, v_{i_{2k}})$?

A *prescribed block* of the prescription in $H_{2k+1,n}$ is a **maximal** subset of consecutive prescribed vertices along its underlying cycle. We distinguish three main cases depending on the number of *large* prescribed blocks in $H_{2k+1,n}$.

1. There is no prescribed block with size at least k.
2. There is exactly one prescribed block with size at least k.
3. There are exactly two prescribed blocks with size k.

In the first two cases, the realization is deduced only thanks to the *cycle edges* of $H_{2k+1,n}$. However, its *diagonal edges* must be used to deal with the third one.

What about optimal AP+2 graphs?

Our proof of previous Theorem 3 does not deal with Harary 3-connected graphs. Actually, these graphs are not all AP+2 because of their weak structure.

Observation

The Harary graph $H_{3,n}$ is not AP+2 when $n \equiv 2 \mod 4$.

Indeed, for a such value of n, a Harary graph $H_{3,n}$ is a *balanced bipartite graph*. It is then easy to find two of its vertices such that the subgraph resulting from their removal does not admit a perfect matching.

On the existence of optimal AP+2 graphs

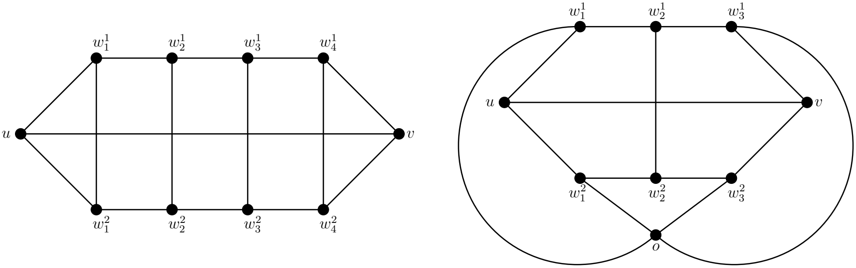
We introduced the following family of graphs.

Definition: *Pr_n graphs*

Let $n \geq 6$. The graph Pr_n is constructed as follows:

- If n is even, Pr_n is obtained from the cycle C_n, whose vertices are successively denoted by $u, w_1^1, ..., w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, ..., w_1^2$, by adding it the edge uv and all edges $w_i^1 w_i^2$, for every $i \in [1, \frac{n-2}{2}]$.
- If n is odd, Pr_n is obtained by first removing the edges $w_1^1 w_1^2$ and $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$ from Pr_{n-1}, and then adding it a new vertex o linked to w_1^1, w_1^2, $w_{\frac{n-3}{2}}^1$, and $w_{\frac{n-3}{2}}^2$.

Examples of Pr_n graphs



The graphs Pr_{10} and Pr_9

Pr_n graphs are optimal AP+2 graphs

Observe that a path can be partitioned under 2-prescriptions (u, v) such that u and v are its endvertices. We thus get the following.

Observation

A Hamiltonian-connected graph is AP+2.

Pr_n graphs are optimal AP+2 graphs

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Observation

A Hamiltonian-connected graph is AP+2.

Regarding Pr_n graphs, this property is easier to check that the one of being AP+2.

Proposition 1 (Baudon, B., Sopena, 2012)

The graph Pr_n is Hamiltonian-connected for every $n \geq 6$.

As a corollary, we finally get the following.

Corollary of Proposition 1

The graph Pr_n is AP+2 for every $n \geq 6$.

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Conclusions

- For every $k \geq 1$ and $n \geq k$, there exist AP+k graphs on n vertices.
- If G is an AP+k graph on n vertices, then

$$\lceil \frac{n(k+1)}{2} \rceil \leq \|G\| \leq \frac{n(n-1)}{2}$$

holds. These bounds are sharp.

Thank you for your attention!