Decomposing graphs into 328 locally irregular graphs

Julien Bensmail[a], Martin Merker[b], Carsten Thomassen[c]

a: Université Nice-Sophia-Antipolis, France
b: Universität Hamburg, Germany
c: Technical University of Denmark, Denmark

LaBRI
October 14th, 2016

General context

Definitions and main notions

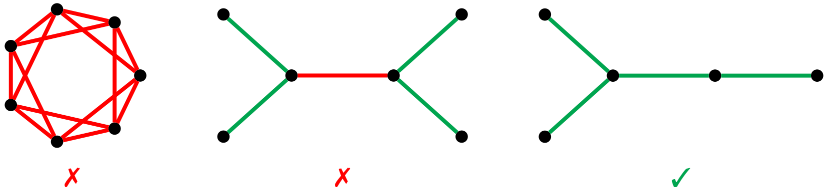
Definitions and main notions

"Un gars" / "Un type" / "Un mec" = A vertex (most of the time)
"Manger" / "Bouffer" / "Piquer" = Move (an edge) from a part to another part

Definitions and main notions

"Un gars" / "Un type" / "Un mec" = A vertex (most of the time)
"Manger" / "Bouffer" / "Piquer" = Move (an edge) from a part to another part

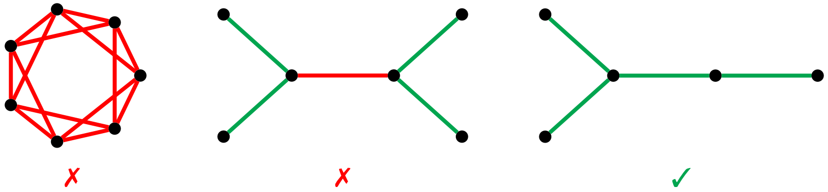
G: undirected graph
G locally irregular = Every two adjacent vertices of G have distinct degrees



Definitions and main notions

"Un gars" / "Un type" / "Un mec" = A vertex (most of the time)
"Manger" / "Bouffer" / "Piquer" = Move (an edge) from a part to another part

G: undirected graph
G locally irregular = Every two adjacent vertices of G have distinct degrees

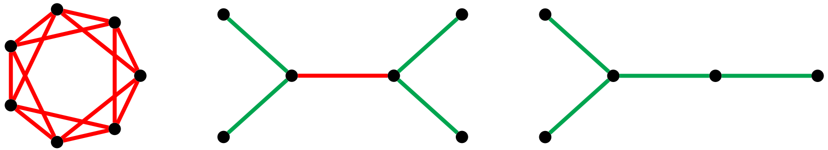


Decomposition of G = Partition $E_1, \ldots, E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, locally irregular edge-colouring)

Definitions and main notions

"Un gars" / "Un type" / "Un mec" = A vertex (most of the time)
"Manger" / "Bouffer" / "Piquer" = Move (an edge) from a part to another part

G: undirected graph
G locally irregular = Every two adjacent vertices of G have distinct degrees



✗ ✗ ✓

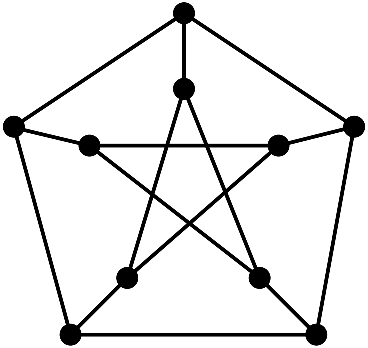
Decomposition of G = Partition $E_1, ..., E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs (equivalently, locally irregular edge-colouring)

$\chi'_{\mathrm{irr}}(G)$ = Smallest $k \geq 1$, s.t. G has locally irregular k-edge-colourings
G decomposable = $\chi'_{\mathrm{irr}}(G)$ exists.
G exceptional, otherwise.

Petersen graph

Theorem

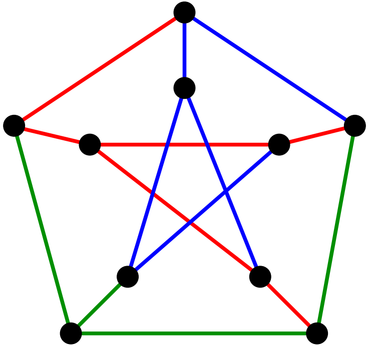
In every GT talk in Bordeaux, one will ask about the Petersen graph.



Petersen graph

Theorem

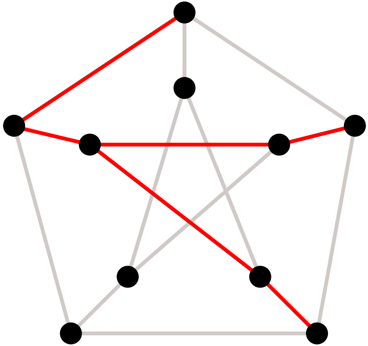
In every GT talk in Bordeaux, one will ask about the Petersen graph.



Petersen graph

Theorem

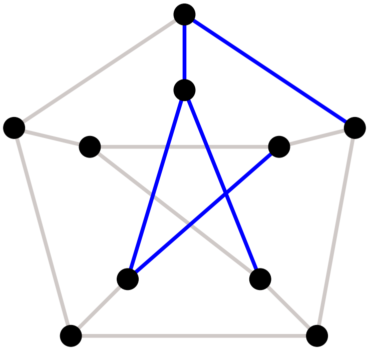
In every GT talk in Bordeaux, one will ask about the Petersen graph.



Petersen graph

Theorem

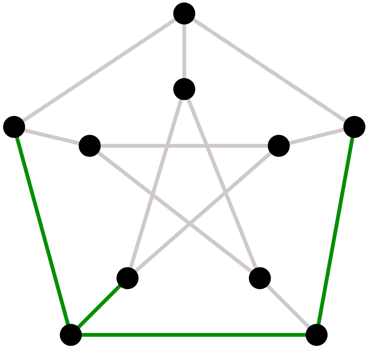
In every GT talk in Bordeaux, one will ask about the Petersen graph.



Petersen graph

Theorem

In every GT talk in Bordeaux, one will ask about the Petersen graph.

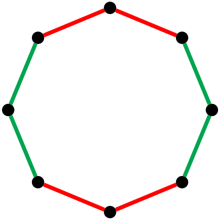


Some motivations

1. Local irregularity = Possible antonym notion to regularity
2. χ'_{irr} = Measure of closeness to irregularity

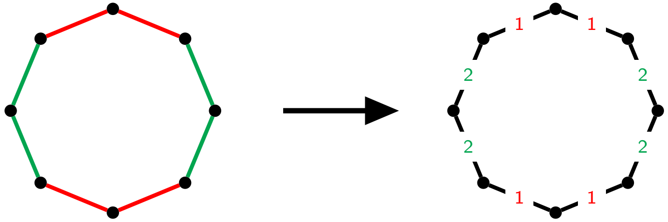
Some motivations

1. Local irregularity = Possible antonym notion to regularity
2. χ'_{irr} = Measure of closeness to irregularity
3. Connexions and applications to the 1-2-3 Conjecture



Some motivations

1. Local irregularity = Possible antonym notion to regularity
2. χ'_{irr} = Measure of closeness to irregularity
3. Connexions and applications to the 1-2-3 Conjecture



Previous works: Exceptional graphs

Exceptional graphs?

Previous works: Exceptional graphs

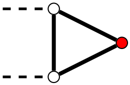
Exceptional graphs?

Some obvious ones: odd-length paths and odd-length cycles...

Previous works: Exceptional graphs

Exceptional graphs?

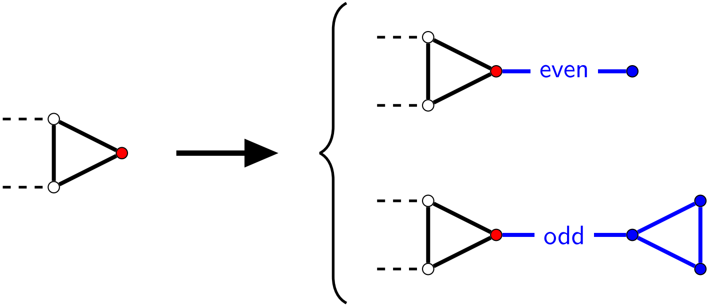
Some obvious ones: odd-length paths and odd-length cycles...
... but also $\mathcal{T}$:



Previous works: Exceptional graphs

Exceptional graphs?

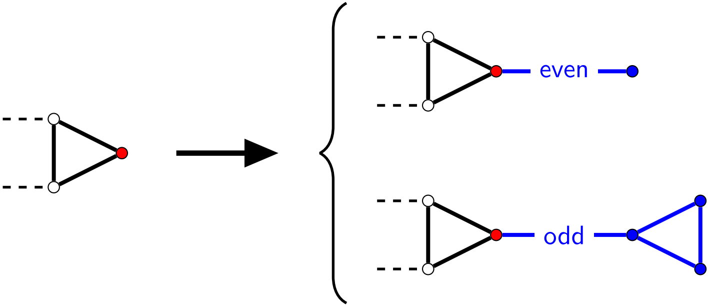
Some obvious ones: odd-length paths and odd-length cycles...
... but also $\mathcal{T}$:



Previous works: Exceptional graphs

Exceptional graphs?

Some obvious ones: odd-length paths and odd-length cycles...
... but also $\mathcal{T}$:



Theorem – Baudon, B., Przybyło, Woźniak (2015)

Exceptional graphs are exactly these three classes of graphs.

Previous works: Main conjecture

How large can χ'_{irr} be?

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2}, K_n, etc.). Actually, unless P = NP, no "good" characterization of when $\chi'_{\mathrm{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2}, K_n, etc.). Actually, unless P = NP, no "good" characterization of when $\chi'_{\mathrm{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Conjecture verified for:

- trees, regular bipartite graphs, $K_{n,m}$, K_n, some Cartesian products, regular graphs with degree $\geq 10^7$ [Baudon, B., Przybyło, Woźniak (2015)]

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2}, K_n, etc.). Actually, unless P = NP, no "good" characterization of when $\chi'_{\mathrm{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Conjecture verified for:

- trees, regular bipartite graphs, $K_{n,m}$, K_n, some Cartesian products, regular graphs with degree $\geq 10^7$ [Baudon, B., Przybyło, Woźniak (2015)]
- graphs with $\delta \geq 10^{10}$ [Przybyło (2016)]

Main questions, and partial answers

Questions:

1. The conjecture for bipartite graphs?
2. General constant upper bounds on χ'_{irr}?

Main questions, and partial answers

Questions:

1. The conjecture for bipartite graphs?
2. General constant upper bounds on χ'_{irr}?

Today's (partial) answers:

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, ..., G_k$ of G, s.t.

- $\chi'_{\mathrm{irr}}(G - (E(G_1) \cup ... \cup E(G_k)))$ is "small"
- $\chi'_{\mathrm{irr}}(G_1), ..., \chi'_{\mathrm{irr}}(G_k)$ are "small"

$\Rightarrow$ Decompose the G_i's and $G - (E(G_1) \cup ... \cup E(G_k))$ independently

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, ..., G_k$ of G, s.t.

- $\chi'_{\mathrm{irr}}(G - (E(G_1) \cup ... \cup E(G_k)))$ is "small"
- $\chi'_{\mathrm{irr}}(G_1), ..., \chi'_{\mathrm{irr}}(G_k)$ are "small"

$\Rightarrow$ Decompose the G_i's and $G - (E(G_1) \cup ... \cup E(G_k))$ independently

Even-size graph $=$ Graph whose all components have even size
Analogously, notion of odd-size graph

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, ..., G_k$ of G, s.t.

- $\chi'_{\mathrm{irr}}(G - (E(G_1) \cup ... \cup E(G_k)))$ is "small"
- $\chi'_{\mathrm{irr}}(G_1), ..., \chi'_{\mathrm{irr}}(G_k)$ are "small"

$\Rightarrow$ Decompose the G_i's and $G - (E(G_1) \cup ... \cup E(G_k))$ independently

Even-size graph $=$ Graph whose all components have even size
Analogously, notion of odd-size graph

Main steps:

1. Reducing the conjecture to even-size graphs

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, \ldots, G_k$ of G, s.t.

- $\chi'_{\mathrm{irr}}(G - (E(G_1) \cup \ldots \cup E(G_k)))$ is "small"
- $\chi'_{\mathrm{irr}}(G_1), \ldots, \chi'_{\mathrm{irr}}(G_k)$ are "small"

$\Rightarrow$ Decompose the G_i's and $G - (E(G_1) \cup \ldots \cup E(G_k))$ independently

Even-size graph = Graph whose all components have even size
Analogously, notion of odd-size graph

Main steps:

1. Reducing the conjecture to even-size graphs
2. Decomposing even-size bipartite graphs

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 10$.
For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs $G_1, ..., G_k$ of G, s.t.

- $\chi'_{\mathrm{irr}}(G - (E(G_1) \cup ... \cup E(G_k)))$ is "small"
- $\chi'_{\mathrm{irr}}(G_1), ..., \chi'_{\mathrm{irr}}(G_k)$ are "small"

$\Rightarrow$ Decompose the G_i's and $G - (E(G_1) \cup ... \cup E(G_k))$ independently

Even-size graph $=$ Graph whose all components have even size
Analogously, notion of odd-size graph

Main steps:

1. Reducing the conjecture to even-size graphs
2. Decomposing even-size bipartite graphs
3. Using Przybyło's and the bipartite results

Step 1: Reducing to even-size graphs

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\mathrm{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\mathrm{irr}}(\text{even-size}) + k$$

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\mathrm{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\mathrm{irr}}(\text{even-size}) + k$$

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\mathrm{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\mathrm{irr}}(\text{even-size}) + k$$

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Corollary

$$\chi'_{\mathrm{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\mathrm{irr}}(\text{even-size}) + 1$$

Two useful easy lemmas

First lemma

Let G be a connected odd-size graph. For every vertex $v \in V(G)$, there exists an edge e incident to v, s.t. $G - e$ is an even-size graph.

Proof. Assume all edges incident to v are cut-edges. Then consider the sizes of the components incident to v. ■

Two useful easy lemmas

First lemma

Let G be a connected odd-size graph. For every vertex $v \in V(G)$, there exists an edge e incident to v, s.t. $G - e$ is an even-size graph.

Proof. Assume all edges incident to v are cut-edges. Then consider the sizes of the components incident to v. ■

Second lemma

Let G be a connected even-size graph. For every vertex $v \in V(G)$, there exists a path P of length 2 containing v, s.t. $G - E(P)$ is an even-size graph.

Proof. Consider any edge uv incident to v, and apply the previous lemma to an odd-size component incident to u or v in $G - uv$. ■

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample.

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;
- Triangles are joined by odd length paths.

Finding a  or  in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+-vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;
- Triangles are joined by odd length paths.

$\Rightarrow G$ is exceptional, a contradiction. ■

Step 2: Decomposing even-size bipartite graphs

Main result, and proof ideas

Theorem

χ'_{irr}(even-size, bipartite) ≤ 9. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

Main result, and proof ideas

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Main result, and proof ideas

Theorem

$\chi'_{\mathrm{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\mathrm{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow$ G locally irregular!

Main result, and proof ideas

Theorem

$\chi'_{\mathrm{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\mathrm{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow$ G locally irregular!

Main idea: Make G as close as possible to this structure, i.e. remove two decomposable subgraphs F_A and F_B with small χ'_{irr}, s.t.

1. all degrees in A get even
2. all degrees in B get odd

Main result, and proof ideas

Theorem

χ'_{irr}(even-size, bipartite) ≤ 9. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow$ G locally irregular!

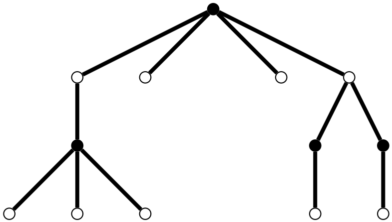
Main idea: Make G as close as possible to this structure, i.e. remove two decomposable subgraphs F_A and F_B with small χ'_{irr}, s.t.

1. all degrees in A get even
2. all degrees in B get odd

$\Rightarrow$ We do that with F_A and F_B being two balanced forests

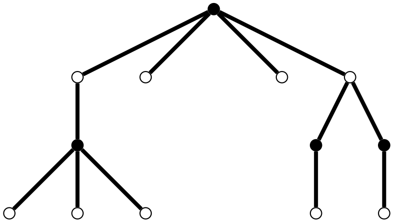
Balanced forests

Balanced forest = In one of the two colour classes, only even degrees



Balanced forests

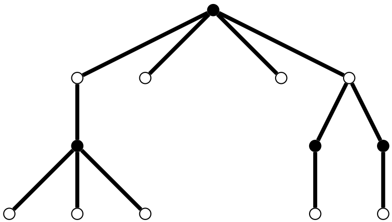
Balanced forest = In one of the two colour classes, only even degrees



Change the degree parities of pairs of vertices of G in the same part?
⇒ Find a balanced forest!

Balanced forests

Balanced forest = In one of the two colour classes, only even degrees



Change the degree parities of pairs of vertices of G in the same part?
$\Rightarrow$ Find a balanced forest!

Lemma

χ'_{irr}(balanced tree) ≤ 2

(though infinitely many trees verify $\chi'_{\mathrm{irr}} = 3$ [Baudon, B., Sopena (2015)])

Getting “almost” the desired degree properties

Reminder: G is an even-size bipartite graph

Getting "almost" the desired degree properties

Reminder: G is an even-size bipartite graph

Theorem

There exists a balanced forest F_A with leaves in A, s.t., in $G - E(F_A)$, all vertices in A have even degree.

Proof. In a spanning forest, take a system F_A of edge-disjoint paths joining pairs of odd-degree vertices in A, which minimizes the total length. ■

Getting "almost" the desired degree properties

Reminder: G is an even-size bipartite graph

Theorem

There exists a balanced forest F_A with leaves in A, s.t., in $G - E(F_A)$, all vertices in A have even degree.

Proof. In a spanning forest, take a system F_A of edge-disjoint paths joining pairs of odd-degree vertices in A, which minimizes the total length. ■

Theorem

There exists a balanced forest F_B with leaves in B, s.t., in $G - E(F_B)$, all vertices of B, but at most one, have odd degree.

Proof. Just do the same, but with a system F_B of edge-disjoint paths joining pairs of even-degree vertices in B. ■

It cannot be that easy...

Note: Degrees in A even $\Rightarrow$ G remains decomposable

It cannot be that easy...

Note: Degrees in A even $\Rightarrow$ G remains decomposable

A remaining even-degree vertex v in B?

It cannot be that easy...

Note: Degrees in A even $\Rightarrow$ G remains decomposable

A remaining even-degree vertex v in B?
$\Rightarrow$ Remove "something" with small χ'_{irr}, so that we get the desired structure

It cannot be that easy...

Note: Degrees in A even $\Rightarrow$ G remains decomposable

A remaining even-degree vertex v in B?
$\Rightarrow$ Remove "something" with small χ'_{irr}, so that we get the desired structure

Case 1. No cycle through v.

$\Rightarrow$ All edges incident to v are cut-edges. Choose any e of them, and let H_1 and H_2 be the two components of $G - e$. Note that $d_{H_1+e}(v)$ and $d_{H_2}(v)$ are odd. Hence $H_1 + e$ and H_2 are locally irregular, and

$$\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(H_1 + e) + \chi'_{\mathrm{irr}}(H_2) + \chi'_{\mathrm{irr}}(F_A) + \chi'_{\mathrm{irr}}(F_B) \leq 6.$$

It cannot be that easy...

Note: Degrees in A even $\Rightarrow$ G remains decomposable

A remaining even-degree vertex v in B?
$\Rightarrow$ Remove "something" with small χ'_{irr}, so that we get the desired structure

Case 1. No cycle through v.

$\Rightarrow$ All edges incident to v are cut-edges. Choose any e of them, and let H_1 and H_2 be the two components of $G - e$. Note that $d_{H_1+e}(v)$ and $d_{H_2}(v)$ are odd. Hence $H_1 + e$ and H_2 are locally irregular, and

$$\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(H_1 + e) + \chi'_{\mathrm{irr}}(H_2) + \chi'_{\mathrm{irr}}(F_A) + \chi'_{\mathrm{irr}}(F_B) \leq 6.$$

Case 2. A cycle C through v.

$\Rightarrow$ Remove C from G; this does not alter the degrees' parities.

So... ?

Lemma

There exists a path P starting at v, s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v. Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move $v_0 v_1$ to P. If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move $v_1 v_2$ to P. Repeat this process until it stops.

So... ?

Lemma

There exists a path P starting at v, s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v. Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move $v_0 v_1$ to P. If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move $v_1 v_2$ to P. Repeat this process until it stops.

Note that the conflicting degrees strictly decrease. Hence all v_i's are different, and the process ends at some point. Furthermore, P induces a path. ■

So… ?

Lemma

There exists a path P starting at v, s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v. Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move $v_0 v_1$ to P. If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move $v_1 v_2$ to P. Repeat this process until it stops.

Note that the conflicting degrees strictly decrease. Hence all v_i's are different, and the process ends at some point. Furthermore, P induces a path. ■

P exceptional?

So... ?

Lemma

There exists a path P starting at v, s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v. Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move $v_0 v_1$ to P. If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move $v_1 v_2$ to P. Repeat this process until it stops.

Note that the conflicting degrees strictly decrease. Hence all v_i's are different, and the process ends at some point. Furthermore, P induces a path. ■

P exceptional? Well, yes, but...

Observation

$$\chi'_{\text{irr}}(C \cup P) \leq 4$$

Gathering everything

Call G' what remains of G;

Gathering everything

Call G' what remains of G; then

$$\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(G') + \chi'_{\mathrm{irr}}(F_A) + \chi'_{\mathrm{irr}}(F_B) + \chi'_{\mathrm{irr}}(C \cup P) \leq 9$$

■

Step 3: Using Przybyło's and the bipartite results

Main result, and rough proof ideas

Theorem

$$\chi'_{\mathrm{irr}}(\text{decomposable}) \leq 328$$

Main result, and rough proof ideas

Theorem

$$\chi'_{\mathrm{irr}}(\text{decomposable}) \leq 328$$

G: decomposable even-size graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\mathrm{irr}}(H) \leq 3$.

Main result, and rough proof ideas

Theorem

$$\chi'_{\mathrm{irr}}(\text{decomposable}) \leq 328$$

G: decomposable even-size graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\mathrm{irr}}(H) \leq 3$.

(Rough) ideas:

1. Decompose G into $H + D$, where:
   - $\delta(H) \geq 10^{10}$
   - D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Main result, and rough proof ideas

Theorem

$$\chi'_{\mathrm{irr}}(\text{decomposable}) \leq 328$$

G: decomposable even-size graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\mathrm{irr}}(H) \leq 3$.

(Rough) ideas:

1. Decompose G into $H + D$, where:
   - $\delta(H) \geq 10^{10}$
   - D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph
2. Decompose D into $\log_2(2 \cdot 10^{10} + 3) + 1$ even-size bipartite graphs

Main result, and rough proof ideas

Theorem

$$\chi'_{\mathrm{irr}}(\text{decomposable}) \leq 328$$

G: decomposable even-size graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\mathrm{irr}}(H) \leq 3$.

(Rough) ideas:

1. Decompose G into $H + D$, where:
   - $\delta(H) \geq 10^{10}$
   - D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph
2. Decompose D into $\log_2(2 \cdot 10^{10} + 3) + 1$ even-size bipartite graphs

$\Rightarrow \chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(H) + \chi'_{\mathrm{irr}}(D) \leq 3 + 9 \cdot 36 = 327$

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H.

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Issue: D might have odd-size components!

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Issue: D might have odd-size components!
Solution: For every component of D, "steal" an incident edge from H

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Issue: D might have odd-size components!
Solution: For every component of D, “steal” an incident edge from H

Issue: A vertex from H might loose a lot of its degree

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Issue: D might have odd-size components!
Solution: For every component of D, "steal" an incident edge from H

Issue: A vertex from H might loose a lot of its degree
Solution: Orient the edges of H in a balanced way, and "steal" out-edges only

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$-degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H. Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$-degenerate.

Issue: D might have odd-size components!
Solution: For every component of D, "steal" an incident edge from H

Issue: A vertex from H might loose a lot of its degree
Solution: Orient the edges of H in a balanced way, and "steal" out-edges only

Note: Eventually, $\delta(H) \geq 10^{10}$ and D remains $(2 \cdot 10^{10} + 2)$-degenerate

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G'. If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G'.

Proof. By induction on d. If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-lengths paths in the bipartite 1- and 2-subgraphs.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G'. If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G'.

Proof. By induction on d. If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-lengths paths in the bipartite 1- and 2-subgraphs.

Note further that, along one such path in the 2-subgraph, it cannot be that every two subsequent vertices are joined by an even-length path in the 1-subgraph. Choose the first pair which does not verify this, and change the colour of the joining edge to 1. Each of the 1- and 2-subgraphs now has one odd-size component, but we can add to them the convenient edge incident to v.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G'. If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G'.

Proof. By induction on d. If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-lengths paths in the bipartite 1- and 2-subgraphs.

Note further that, along one such path in the 2-subgraph, it cannot be that every two subsequent vertices are joined by an even-length path in the 1-subgraph. Choose the first pair which does not verify this, and change the colour of the joining edge to 1. Each of the 1- and 2-subgraphs now has one odd-size component, but we can add to them the convenient edge incident to v.

For the induction step, we consider $G' - v$, and repeatedly consider one of the $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, and add as many edges to them. This number is about $\frac{d(v)}{2}$, $\frac{d(v)}{4}$, and so on. ∎

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a even-size d-degenerate graph. Then G' can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a even-size d-degenerate graph. Then G' can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Let v be a vertex with smallest degree different from 1. If $d(v)$ is even, just apply the previous lemma. Consider thus the case where $d(v)$ is odd.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a even-size d-degenerate graph. Then G' can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Let v be a vertex with smallest degree different from 1. If $d(v)$ is even, just apply the previous lemma. Consider thus the case where $d(v)$ is odd.

We consider $G' - v$. If there is no isolated vertex w, we add such a vertex w, and an edge joining w and another neighbour u of v. So, $G' - v + uw$ has even size, is d-degenerate, and hence admits a decomposition into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs. In G that decomposition is good, except that there is an odd-size bipartite graph containing u. Then just add a “large” odd number of edges incident to v to that odd-size bipartite graph. It remains an even number of edges incident to v, that we can add to the other even-size bipartite graphs. ■

Conclusion and perspectives

Conclusion and perspectivs

1. Improving our proof scheme?
   - Not using Przybyło's result?
   - Using other kinds of auxiliary decompositions?

Conclusion and perspectivs

1. Improving our proof scheme?
   - Not using Przybyło's result?
   - Using other kinds of auxiliary decompositions?
2. Bipartite graphs?
   - χ'_{irr}(path+cycle) ≤ 3?
   - Other method not based on even-odd degrees?

Conclusion and perspectivs

1. Improving our proof scheme?
   - Not using Przybyło's result?
   - Using other kinds of auxiliary decompositions?
2. Bipartite graphs?
   - χ'_{irr}(path+cycle) ≤ 3?
   - Other method not based on even-odd degrees?
3. Other classes of graphs?
   - Bounded degree?
   - Sparse classes?

Conclusion and perspectivs

1. Improving our proof scheme?
   - Not using Przybyło's result?
   - Using other kinds of auxiliary decompositions?
2. Bipartite graphs?
   - $\chi'_{\mathrm{irr}}(\text{path+cycle}) \leq 3$?
   - Other method not based on even-odd degrees?
3. Other classes of graphs?
   - Bounded degree?
   - Sparse classes?
4. When does $\chi'_{\mathrm{irr}}(G) \leq 2$ hold?
   - Bipartite graphs?
   - Large degree?

Conclusion and perspectivs

1. Improving our proof scheme?
   - Not using Przybyło's result?
   - Using other kinds of auxiliary decompositions?
2. Bipartite graphs?
   - χ'_{irr}(path+cycle) ≤ 3?
   - Other method not based on even-odd degrees?
3. Other classes of graphs?
   - Bounded degree?
   - Sparse classes?
4. When does $\chi'_{\mathrm{irr}}(G) \leq 2$ hold?
   - Bipartite graphs?
   - Large degree?

Thank you for your attention!