On the "quest" towards a directed variant of the 1-2-3 Conjecture

Julien Bensmail* (w/ many others)

*Université Côte d'Azur, France

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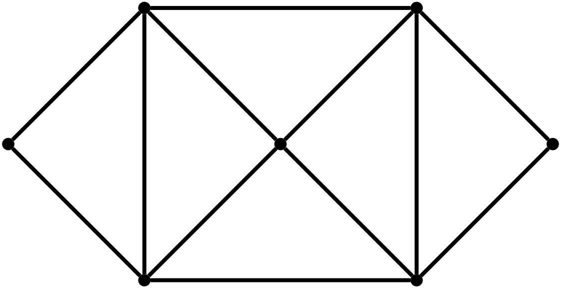
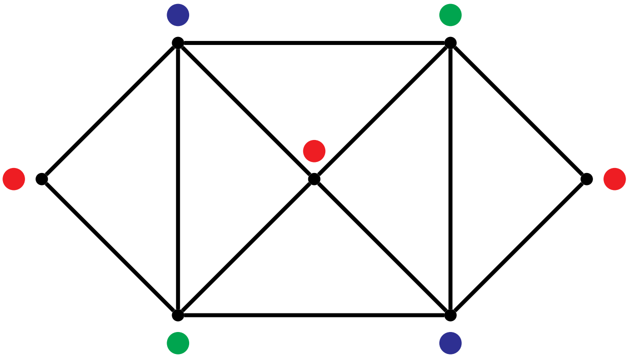
September 11, 2019

General introduction

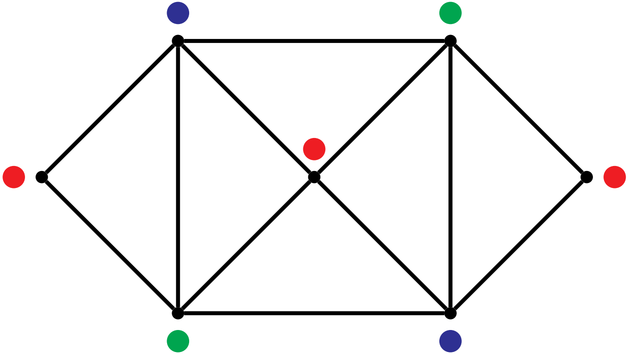
Make adjacent vertices distinguishable?



Make adjacent vertices distinguishable? ⇒ Proper vertex-colouring ☺



Make adjacent vertices distinguishable? $\Rightarrow$ Proper vertex-colouring ☺



⚠ χ might be as high as $\Delta+1$ (Brooks' Theorem)

"Encode" a proper vertex-colouring using few different types of resources?

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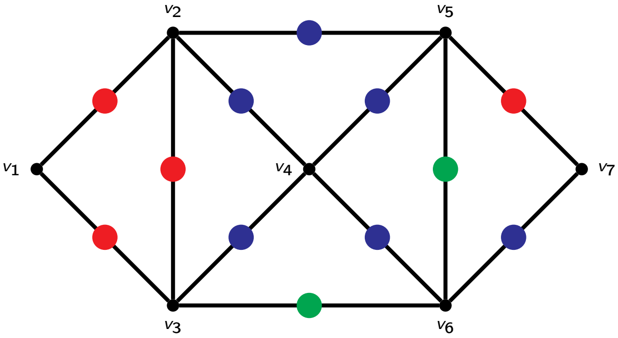

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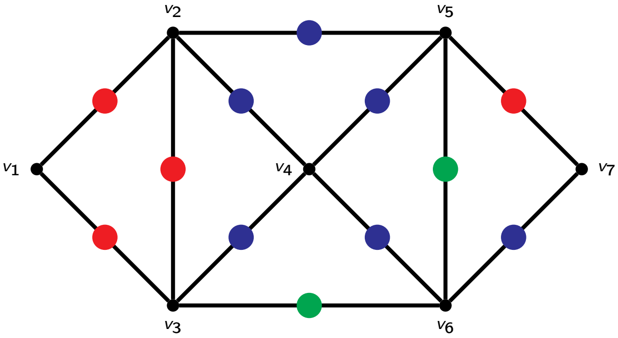

$\mathrm{Col}(v_i)$:= Set of colours "incident" to v_i:

$$\mathrm{Col}(v_1) = \{{\color{red}\bullet}\} \quad \mathrm{Col}(v_2) = \{{\color{red}\bullet}, {\color{blue}\bullet}\} \quad \mathrm{Col}(v_3) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$$

$$\mathrm{Col}(v_4) = \{{\color{blue}\bullet}\} \quad \mathrm{Col}(v_5) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_6) = \{{\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_7) = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$$

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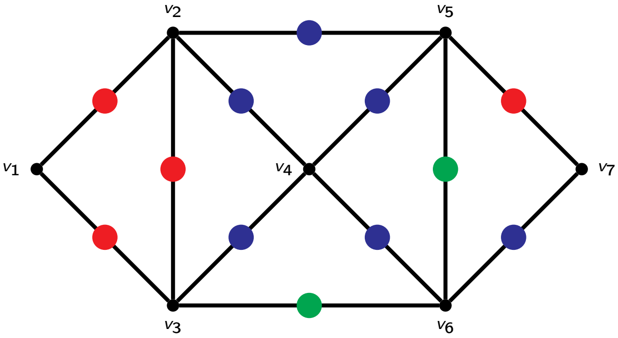

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Neighbours are distinguished!

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From edge colours to vertex colours

Many parameters:

- How is Col computed?
- Edge colour restrictions (properness, etc.)?
- What elements are coloured (edges, vertices, both, etc.)?
- etc.

⇒ Dozens and dozens variants...

A Dynamic Survey of Graph Labeling

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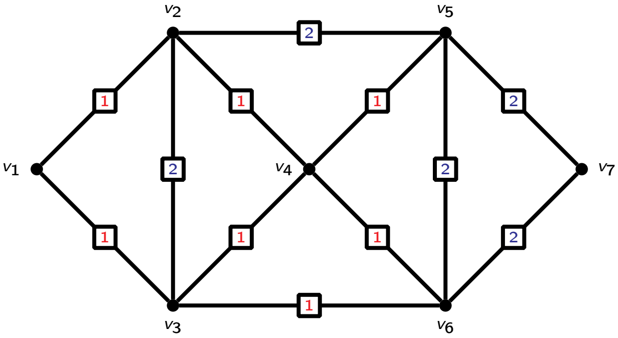
Submitted: September 1, 1996; Accepted: November 14, 1997
Twentieth edition, December 22, 2017
Mathematics Subject Classifications: 05C78

Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 2500 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

1-2-3 Conjecture
– Introduction –

Edge-colours = Edge-weights
$\mathrm{Col}(v_i) = \sigma(v_i)$:= Sums of weights "incident" to v_i



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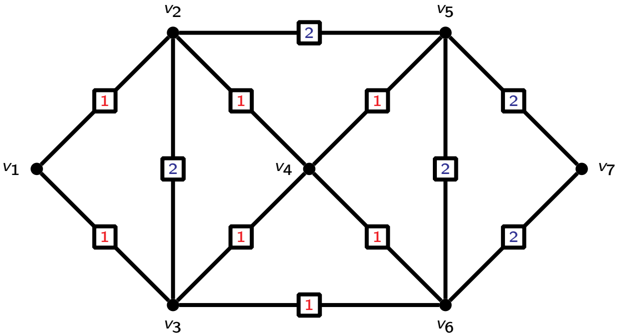

$\sigma(v_1) = 2 \quad \sigma(v_2) = 6 \quad \sigma(v_3) = 5 \quad \sigma(v_4) = 4$
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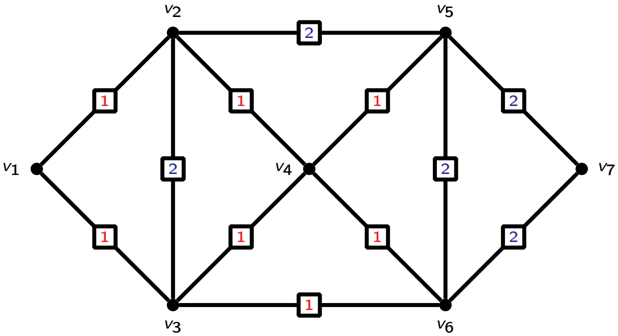

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$\chi^e_\Sigma = 2$ while $\chi = 3$ ☺

Neighbour-sum-distinguishing edge-weighting = σ is proper
$\chi^e_\Sigma(G)$ = smallest k such that G has n-s-d k-edge-weightings

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Nice graph = no K_2 as a component

Neighbour-sum-distinguishing edge-weighting $= \sigma$ is proper
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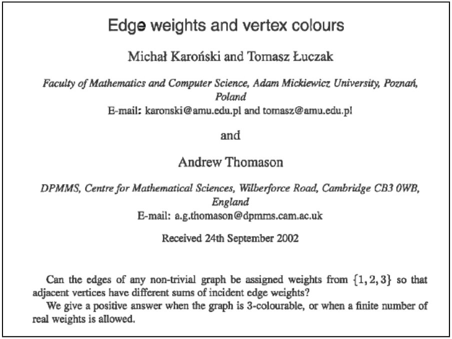
1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

For every nice graph G, we have $\chi^e_\Sigma(G) \leq 3$.



Edge weights and vertex colours

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E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
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Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?
We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

1-2-3 Conjecture

– Some families of graphs –

Theorem

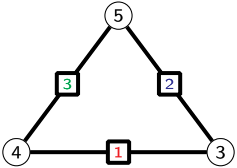
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Make a guess ☺

Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. By induction on n. For $n = 3$:



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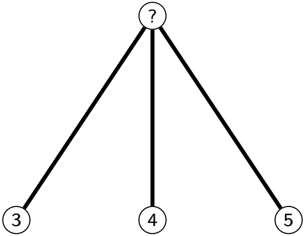
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. $n = 4$:



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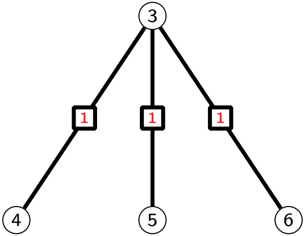
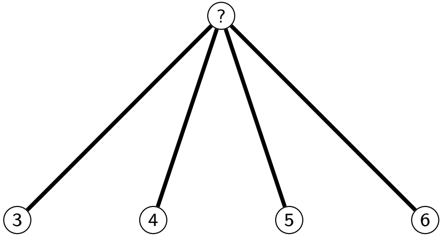


Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. $n = 5$:



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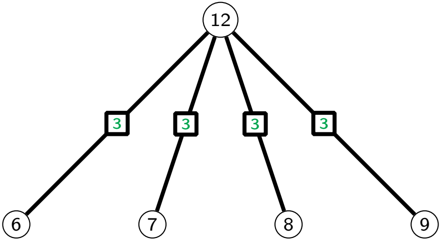

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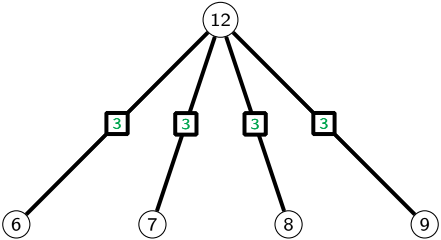

General case: n even $\Rightarrow$ 1's. n odd $\Rightarrow$ 3's.

■

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Any idea ☺ ?

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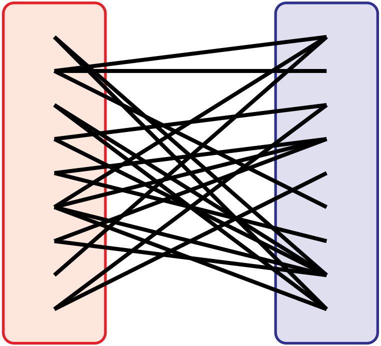
Proof. Bipartition (A, B)

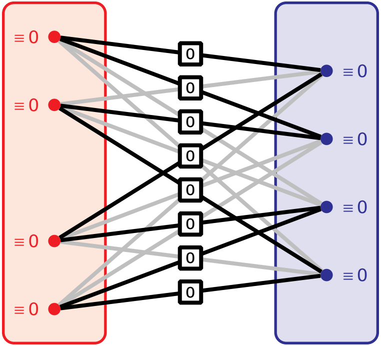


Aim: 3-edge-weighting where $\sigma(A) \equiv 1,2 \pmod 3$ and $\sigma(B) \equiv 0 \pmod 3$
$\Leftrightarrow$ $\{0,1,2\}$-edge-weighting with the same properties

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Proof. Assume $|A|$ is even. Start with weights 0. Second condition fulfilled by B.



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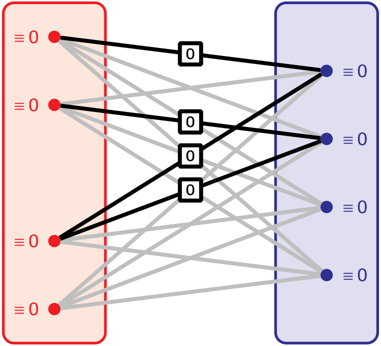
For every nice bipartite graph G, we have $\chi^e_\Sigma(G) \leq 3$.

Proof. Pick a path from A to A with new ends, and apply $+1, -1, \ldots$ along



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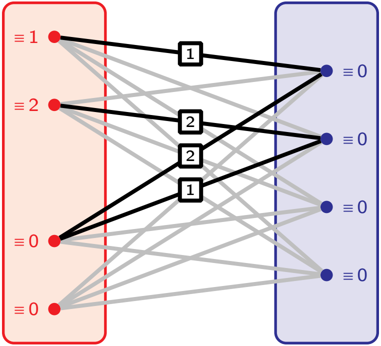

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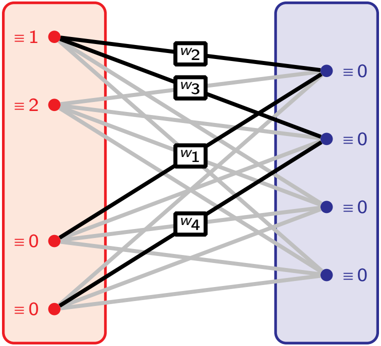
Proof. Repeat until A fulfils the first condition



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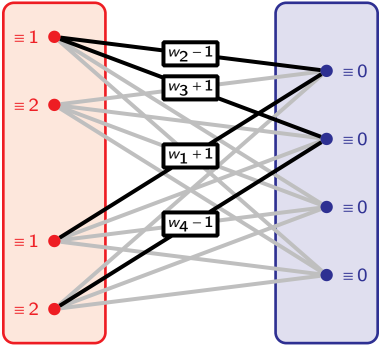

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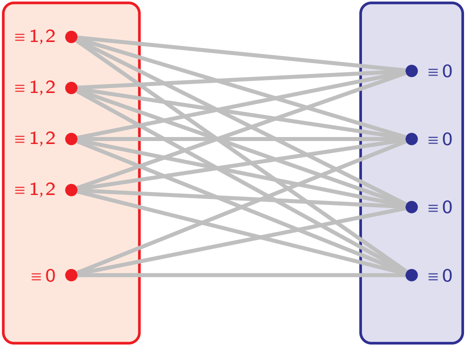
Proof. If $|A|$ and $|B|$ are odd ☹ ... but can reach:



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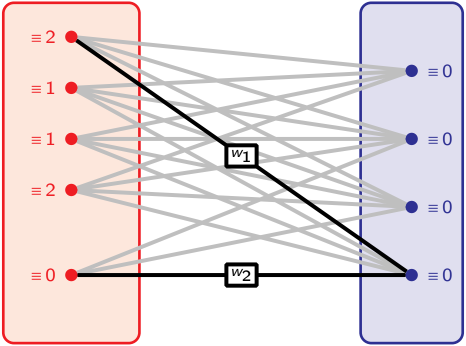
Proof. Eventually apply $+1, -1, \ldots$ or conversely towards another vertex in A



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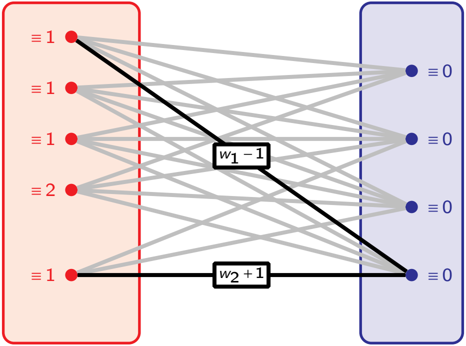

- Proof applies to 3-chromatic graphs with partite sets A, B, C:
  - Use weights $0, 1, 2$
  - Aim $\sigma(A) \equiv 0 \pmod 3$, $\sigma(B) \equiv 1 \pmod 3$, $\sigma(C) \equiv 2 \pmod 3$

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- More generally, k-chromatic graphs, $k \geq 3$ odd, with partite sets $S_0, \ldots, S_{k-1}$:
  - Use weights $0, \ldots, k-1$
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  - Use weights $0, \ldots, k-1$
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- k-chromatic graphs, $k \geq 4$ even, same trick as bipartite graphs

- In general, using $\{1, 2, 3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi_{\Sigma}^{e} \leq 2$ is NP-complete [Dudek, Wajc, 2011]

- In general, using $\{1,2,3\}$ is best possible!
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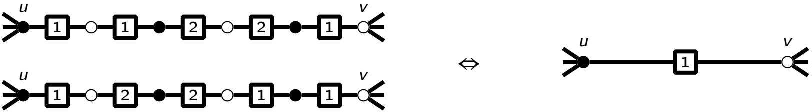
- **Q.:** Is this true for bipartite graphs?
  A.: $\chi^e_\Sigma(\text{Bipartite}) = 3$: *odd multicacti* [Thomassen, Wu, Zhang, 2016]

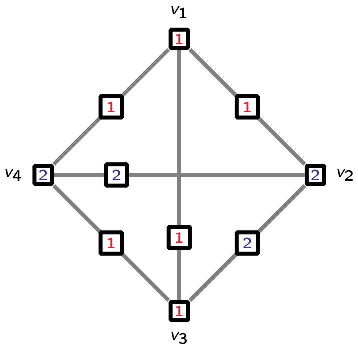
> These graphs can also be described in another way as follows. Take a collection of simple cycles each of length 2 modulo 4 and each with edges colored alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity ≥ 1. The graph with one edge and two vertices is also called an odd multi-cactus.

Intuition: Essentially, with $\{1,2\}$, paths of length $\equiv 1 \pmod 4$ act as edges:

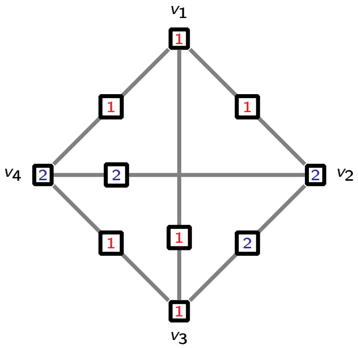


1-2-3 Conjecture
– Best bound –

Best bound on χ^e_Σ obtained from one for a total variant of the problem



Best bound on χ_Σ^e obtained from one for a total variant of the problem



$\sigma(v_1) = 4 \qquad \sigma(v_2) = 7 \qquad \sigma(v_3) = 5 \qquad \sigma(v_4) = 6$

($\sim$ adding a loop at each vertex)
$\chi_\Sigma^t(G)$ = smallest k such that G has n-s-d k-total-weightings

Remarks:

- $\chi^t_\Sigma(G)$ defined for all G
- $\chi^t_\Sigma(G) \leq \chi^e_\Sigma(G)$ for every G

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On a 1, 2 Conjecture

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received February 12, 2008, accepted February 3, 2010.

Let us assign positive integers to the edges and vertices of a simple graph G. As a result we obtain a vertex-colouring of G with integers, where a vertex colour is simply a sum of the weight assigned to the vertex itself and the weights of its incident edges. Can we obtain a proper colouring using only weights 1 and 2 for an arbitrary G?

We give a positive answer when G is a 3-colourable, complete or 4-regular graph. We also show that it is enough to use weights from 1 to 11, as well as from 1 to $\lfloor \frac{\chi(G)}{2} \rfloor + 1$, for an arbitrary graph G.

Keywords: neighbour-distinguishing total-weighting, irregularity strength

1-2 Conjecture [Przybyło, Woźniak, 2010]

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  - Choose a set $\{\phi(v_i), \phi(v_i)+1\}$ of possible sums
    ($\phi(v_i)+1$ = eventual sum, $\phi(v_i)$ the only allowed different sum)
    ⚠ Make sure that $\phi(v_i) \neq \phi(v_j)$ for every backward edge!

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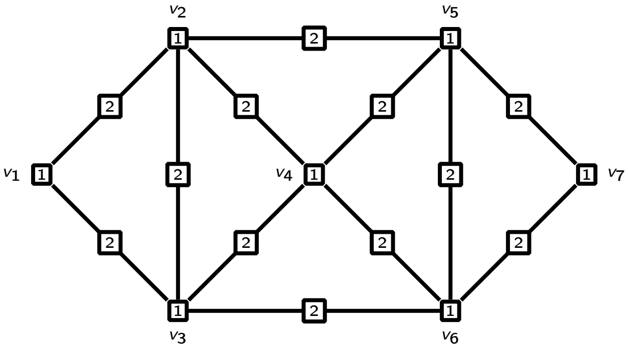
  - Make "valid" weight changes backwards so that $\sigma(v_i) \in \{\phi(v_i), \phi(v_i)+1\}$
- Eventually, do +1 on every vertex weight where $\sigma(v_i) = \phi(v_i)$

Note: Actually, only $1, 2$ are used as vertex weights

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



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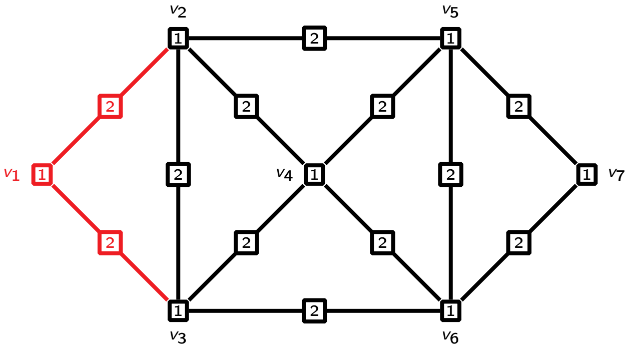

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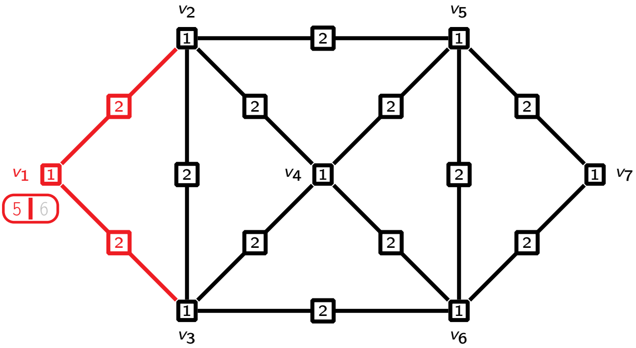

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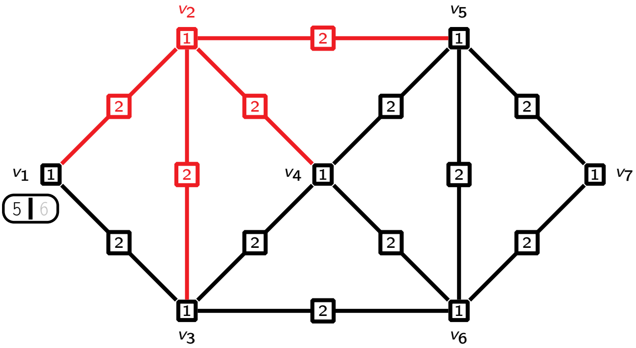
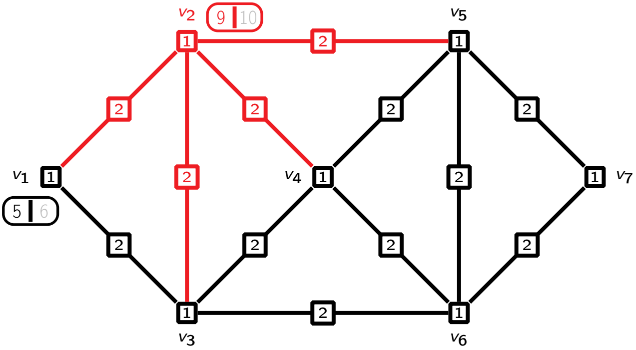


Vertex ordering: $v_1, {\color{red}v_2}, v_3, v_4, v_5, v_6, v_7$



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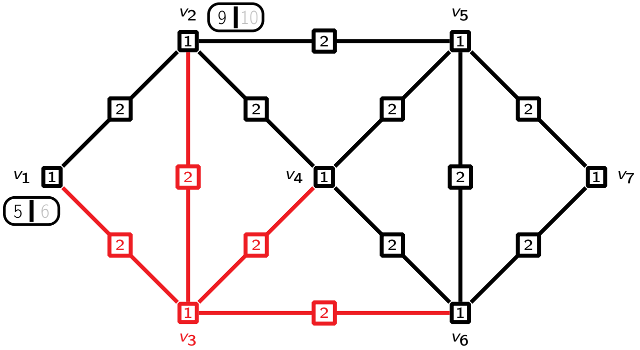

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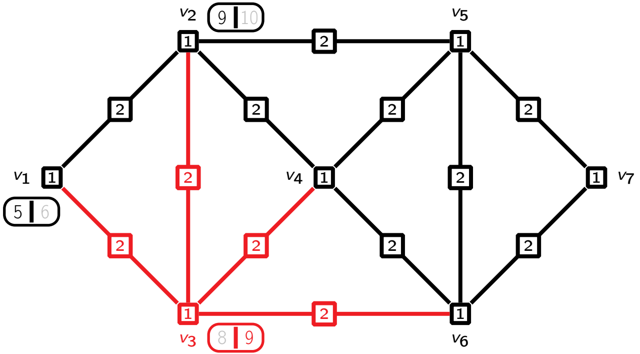

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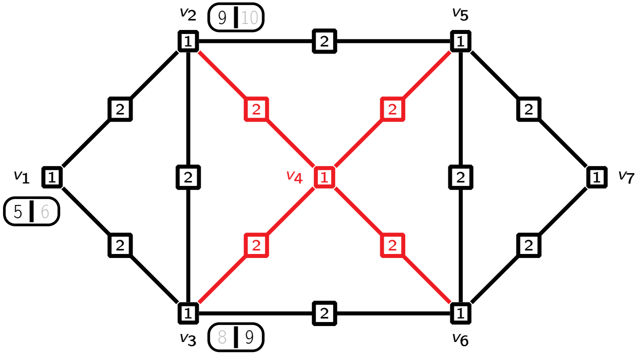

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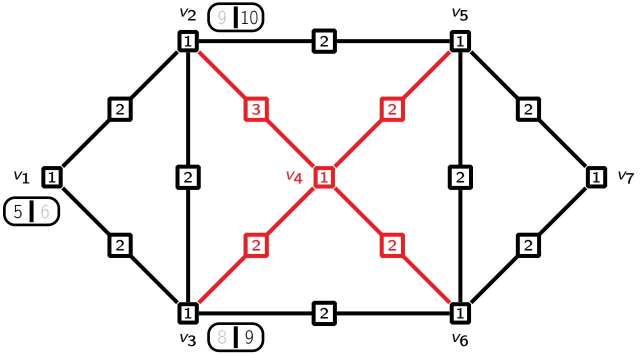

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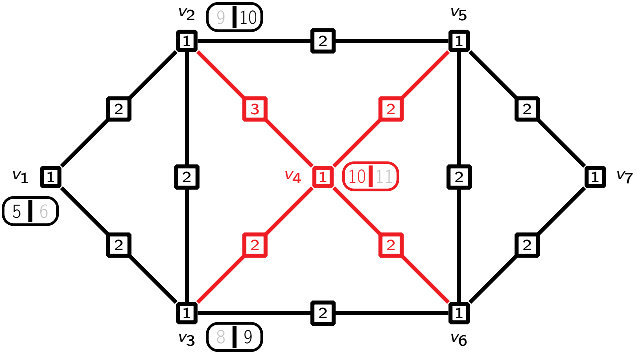

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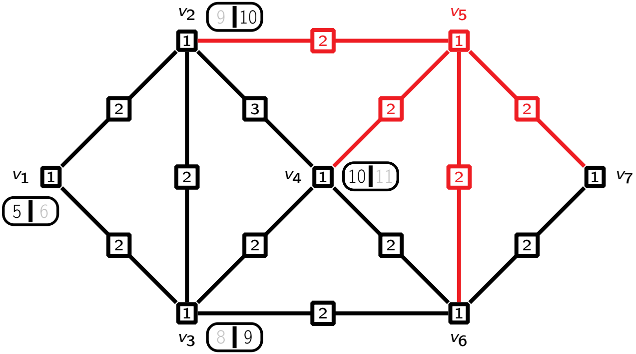

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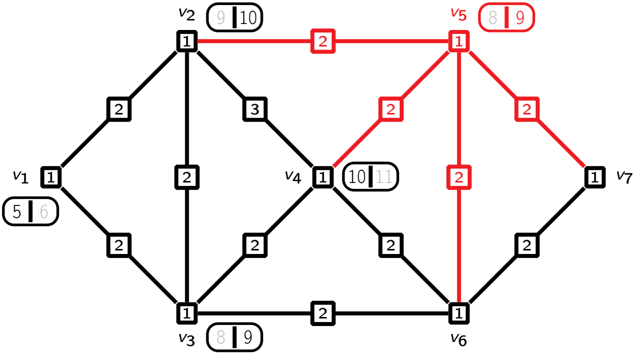

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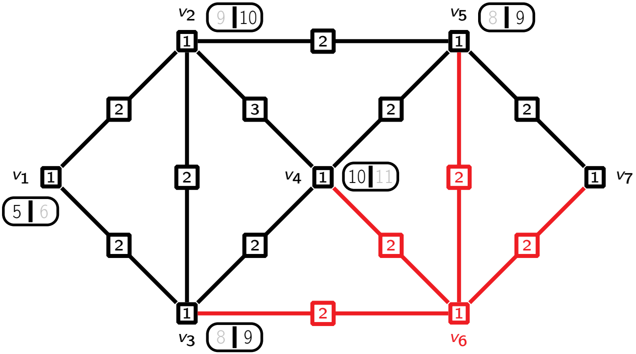

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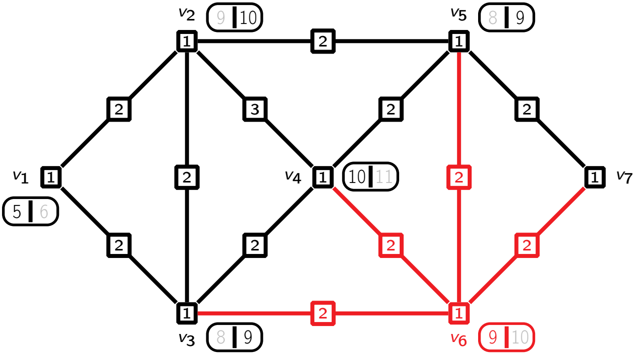

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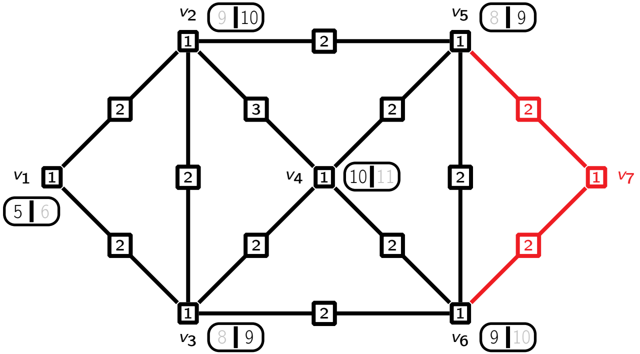

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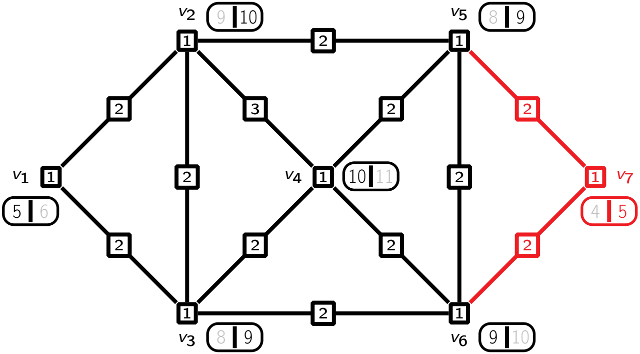
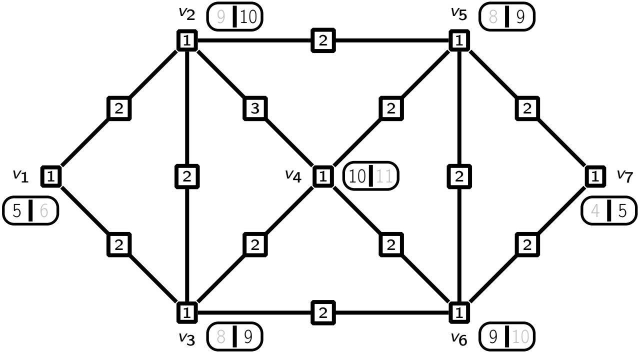
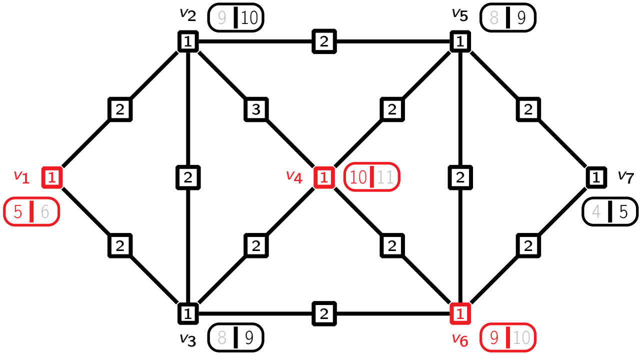




v_2
9 | 10
v_5
8 | 9
v_1
5 | 6
v_4
10 | 11
v_7
4 | 5
v_3
8 | 9
v_6
9 | 10


v2
9 | 10
v5
8 | 9
1
2
1
2
3
2
2
v1
1
5 | 6
2
v4
1
10 | 11
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v7
4 | 5
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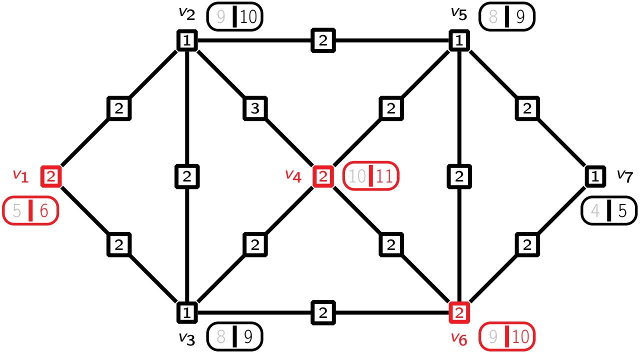
2
1
v3
8 | 9
v6
9 | 10


v2
9 10
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v5
8 9
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v1
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v4
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v7
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4 5
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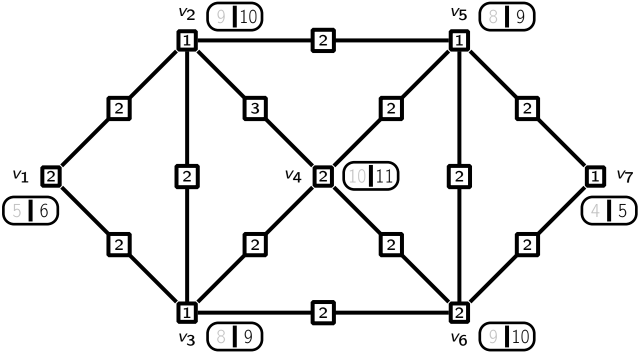
v3
8 9
v6
9 10


v2
9
10
v5
8
9
1
2
1
2
3
2
2
v1
2
2
v4
2
10
11
2
1
v7
5
6
4
5
2
2
2
2
1
2
2
v3
8
9
v6
9
10

- Works because:
  - All edge weight changes are done backwards
  - ⇒ When treating v_i, every backward edge $v_j v_i$ is weighted 2
  - ⇒ A valid change (-1 or +1) per backward edge
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  - ⚠ Valid changes backwards are trickier

1-2-3 Conjecture
– Open questions –

- Prove the 1-2-3 Conjecture for 4-chromatic graphs
  - Done for 4-edge-connected 4-chromatic graphs [Wu, Zhang, Zhu, 2017]

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- Prove/Disprove the 1-2 Conjecture
- List variants?
  - Every graph is (2,3)-choosable [Wong, Zhu, 2016]
  - No constant bound for the edge version ☹

Going to digraphs

Going to Digraphs

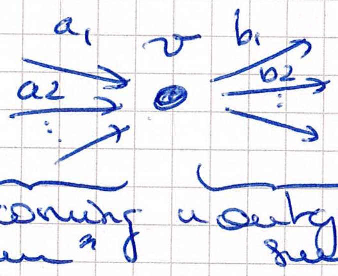
(I)

Wonder how the 123C generalizes to digraphs. Would like some challenge.

In particular:

- same effect when weighting arcs.
- Inductive arguments hard to work out.
- etc..

~~Of course~~ Several options, because "two sums" for the vertices:



Of course, asking $\sigma^- + \sigma^+ \neq \sigma^- + \sigma^+$ is just the 123 Conjecture :)

First work: "Relative sums" — Borowiecki, Grytczuk, Pilśniak 2012

$|\sigma^- - \sigma^+| \neq |\sigma^- - \sigma^+|$

1,2 suffice. Also list version — Khatirinejad, Naserasr, Newman, Seamone, Stevens 2011.

Here, focus on disting. one of $\sigma^-(u), \sigma^+(u)$ and one of $\sigma^-(v), \sigma^+(v)$

(1) $(+,+)$ $u \to v$ $\sigma^+(u) \neq \sigma^+(v)$ — Baudon, B, Sopena 2015

all digraphs can be weighted

Sometimes need 1,2,3

1,2,3 works! induction

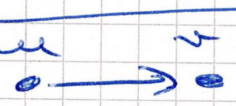
choose v st $d^+(v) \geq d^-(v)$
exists since $\sum d^- = \sum d^+$

Key fact: weighting $\vec{v}$ only affects $\sigma^+(v)$. Can reach sums in $d^+, \dots, 3d^+$, which is $3d^+ - d^+ + 1 = 2d^+ + 1$ values so there is one avoiding that of the neighb ∎

And deciding the index ≤ 2 is NPC.

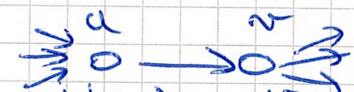
$(-,-)$ same up to reversing the arcs.

~~2017 Wong, P[illegible]. Barme B~~

$(+,-)$ 

Note: "true behaviour" when weighting an arc.

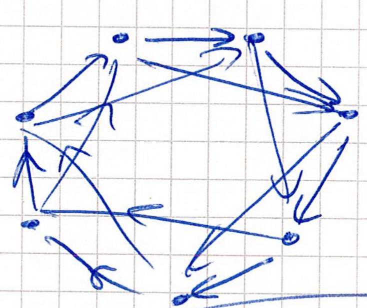
Not all arcs can be weighted: see lonely arcs:



$d^+(u) = d^-(v) = 1$

but ok if no LA.

Note: Sometimes '3':



1,2,3 actually suffice!

Proof: Bip graph $B(D)$ associated to D: $V^- \cup V^+$

- Explode vertices v to v^-, v^+
- $\forall\, u \to v$, add edge u^+v^-!

$\Rightarrow$ Equivalence between weighting D and $B(D)$ 'cause $\sigma^+(v) = \sigma(v^+)$ & $\sigma^-(v) = \sigma(v^-)$

Since the 1-2-3 C holds for bip... ∎

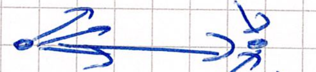
Note: shows that index ≤ 2 iff $B(D)$ is a bip. graph w/ $\chi_\Sigma \leq 2$. Can be checked in polytime due to a characterization (Thomassen, Wu, Zhang).

$(-,+)$:   Horňák Przybyło Woźniak   2018   (3)

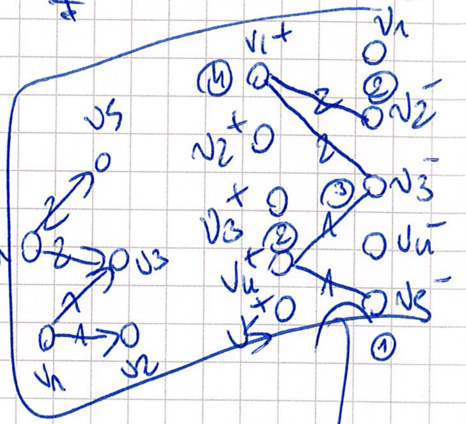
Weirdest variant 'cause weighting $u \to v$ does not affect the sums of interest...

Not defined for all digraphs: ∄

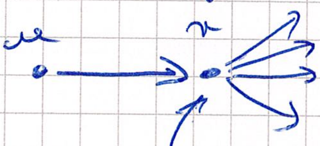
SS arc



$\sigma^- = 0 = \sigma^+$



Also not bounded if ∠A :



σ only determined by $u \to v$...

But otherwise, conjecture of 1,2,3. Sometimes need 3.

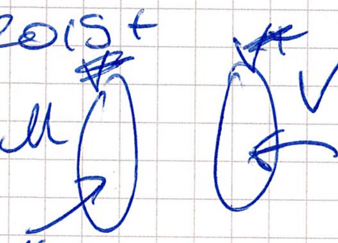
Theorem HPW : 4 work.

Idea use B(D) again.
Good in terms of sums...
... but does not depict the constraints...

Solution : Regardless of the constraints weight B(D) so that no vertex of V^- has the same sum as in V^+.

1,3 ← 2,4 odd...

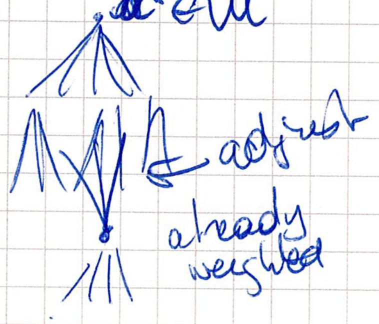
Proof of 1,2,3 : w/ Kasper 2015+
same but refined sum...

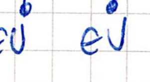


0,3,
4,7,10,13,16...

0,1,2
5,6
8,9
11,12
14,15...

Done by RevBFS...
rooted at $u \in M$
deg ≥ 2

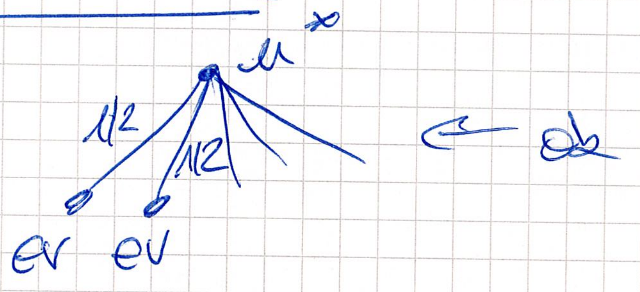


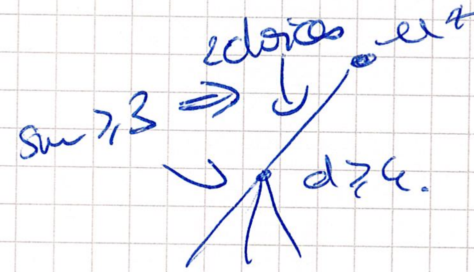


Pbm : u^* is "forced"

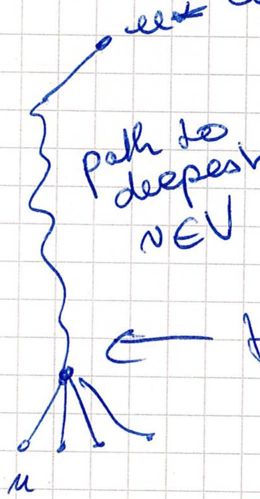
Sol : Show that choices at the last edges.

For instance :





in other cases, no quite choice...



← for that guy, choices. Proof can make these choices go upwards.

And just do that w/ two paths. ☒

Also, E2 is NPC

CCL: Really nice that such different behaviours, different complexity, and proof arguments..
– but deceiving regarding having a challenging pblm in digraphs...