On the "quest" towards a directed variant of the 1-2-3 Conjecture

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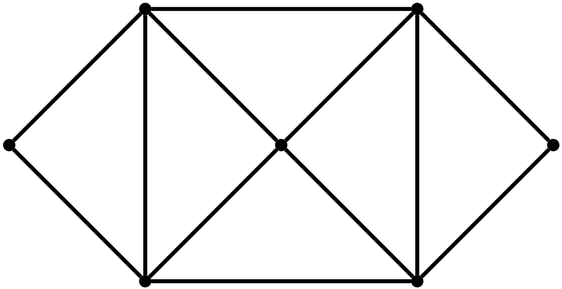
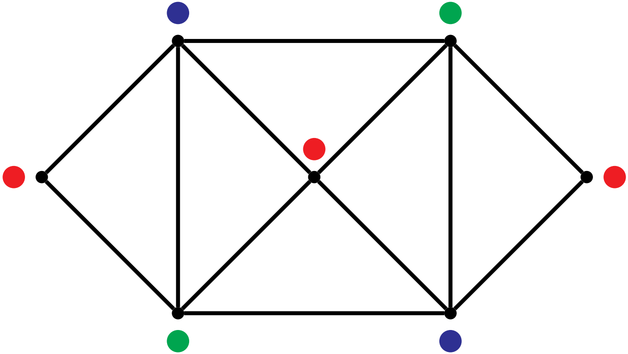
October 28, 2019

General introduction

Make adjacent vertices distinguishable?



Make adjacent vertices distinguishable? ⇒ Proper vertex-colouring ☺



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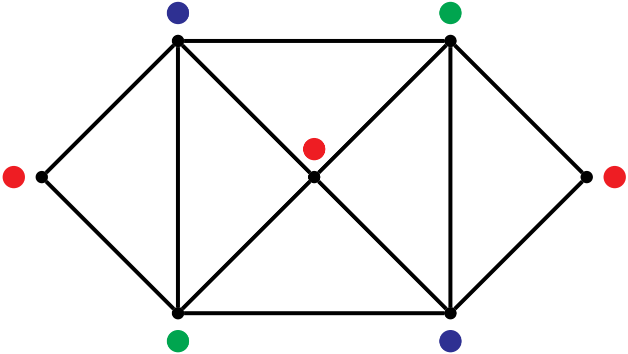

⚠ χ might be as high as $\Delta+1$ (Brooks' Theorem)

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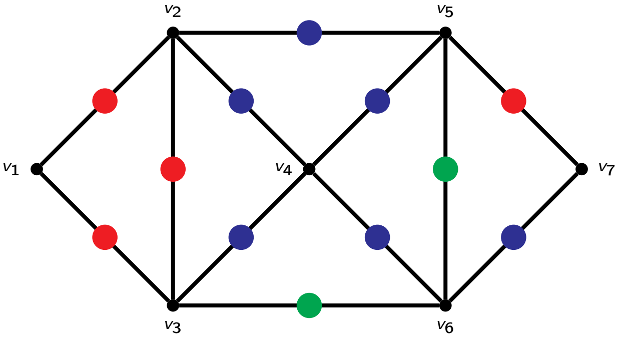

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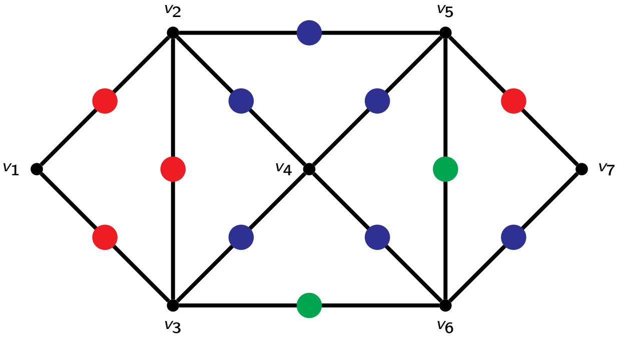

$\mathrm{Col}(v_i)$:= Set of colours "incident" to v_i:

$$\mathrm{Col}(v_1) = \{{\color{red}\bullet}\} \qquad \mathrm{Col}(v_2) = \{{\color{red}\bullet}, {\color{blue}\bullet}\} \qquad \mathrm{Col}(v_3) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$$
$$\mathrm{Col}(v_4) = \{{\color{blue}\bullet}\} \quad \mathrm{Col}(v_5) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_6) = \{{\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_7) = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$$

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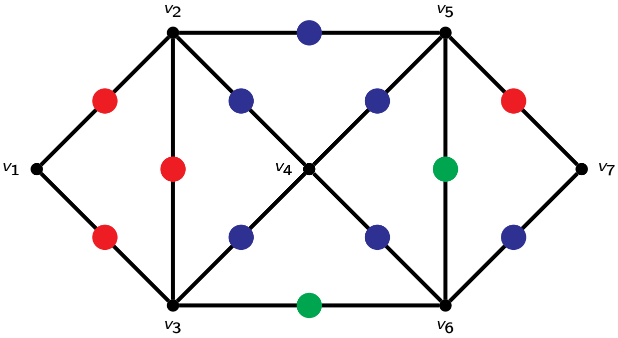

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Neighbours are distinguished!

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- etc.

⇒ Dozens and dozens variants...

A Dynamic Survey of Graph Labeling

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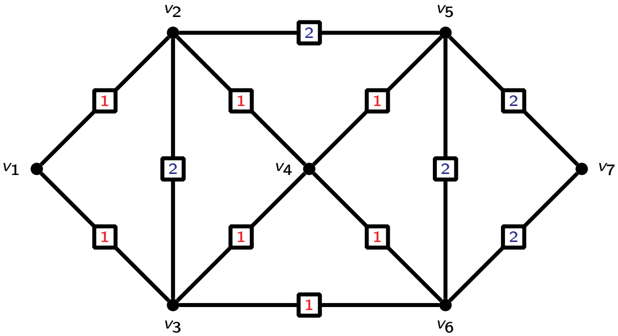
Submitted: September 1, 1996; Accepted: November 14, 1997
Twentieth edition, December 22, 2017
Mathematics Subject Classifications: 05C78

Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 2500 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

1-2-3 Conjecture
– Introduction –

Edge-colours = Edge-weights
$\mathrm{Col}(v_i) = \sigma(v_i) :=$ Sums of weights "incident" to v_i



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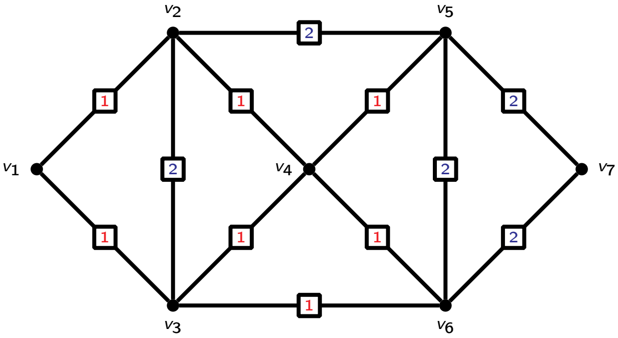

$\sigma(v_1) = 2 \quad \sigma(v_2) = 6 \quad \sigma(v_3) = 5 \quad \sigma(v_4) = 4$
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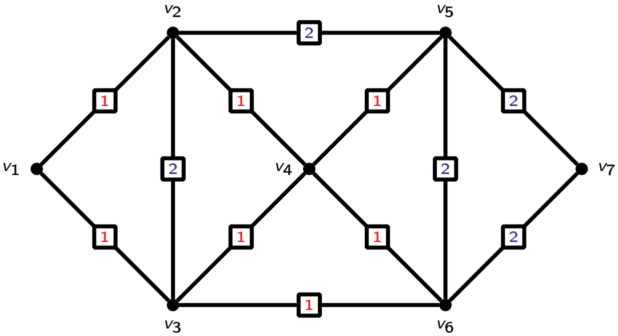

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$\chi^e_\Sigma = 2$ while $\chi = 3$ ☺

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$\chi^e_\Sigma(G)$ = smallest k such that G has n-s-d k-edge-weightings

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Nice graph = no K_2 as a component

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1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

For every nice graph G, we have $\chi^e_\Sigma(G) \leq 3$.

Edge weights and vertex colours

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E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
E-mail: a.g.thomason@dpmms.cam.ac.uk

Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?
We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

1-2-3 Conjecture

– Some families of graphs –

Theorem

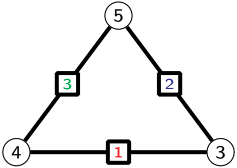
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Make a guess ☺

Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

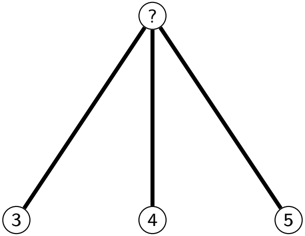
Proof. By induction on n. For $n = 3$:



Theorem

For every $n \geq 3$, we have $\chi_{\Sigma}^{e}(K_n) = 3$.

Proof. $n = 4$:



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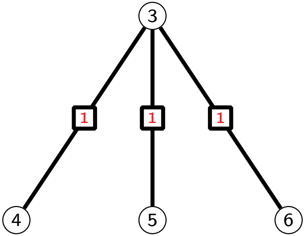
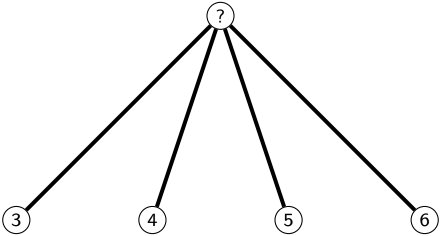


Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. $n = 5$:



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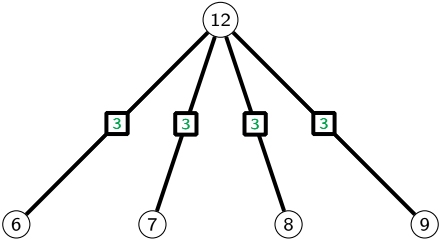

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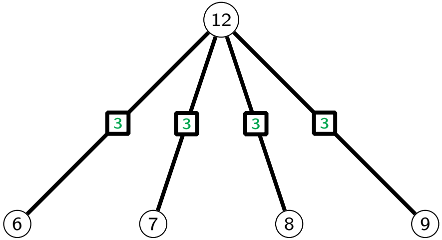

General case: n even $\Rightarrow$ 1's. n odd $\Rightarrow$ 3's.

∎

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Any idea ☺ ?

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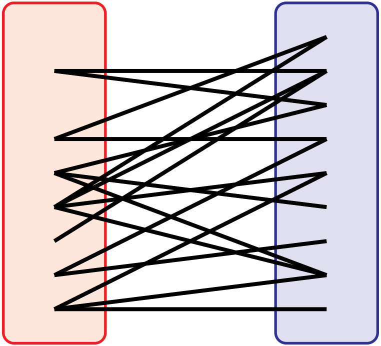
Proof. Bipartition (A, B)

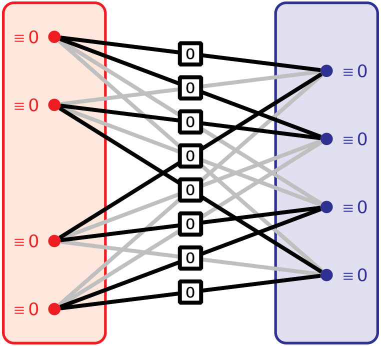


Aim: 3-edge-weighting where $\sigma(A) \equiv 1,2 \pmod 3$ and $\sigma(B) \equiv 0 \pmod 3$
$\Leftrightarrow$ $\{0,1,2\}$-edge-weighting with the same properties

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Proof. Assume $|A|$ is even. Start with weights 0. Second condition fulfilled by B.



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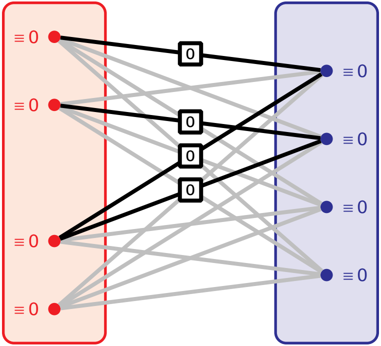
Proof. Pick a path from A to A with new ends, and apply $+1, -1, \ldots$ along



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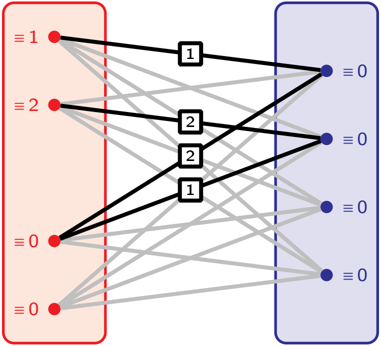

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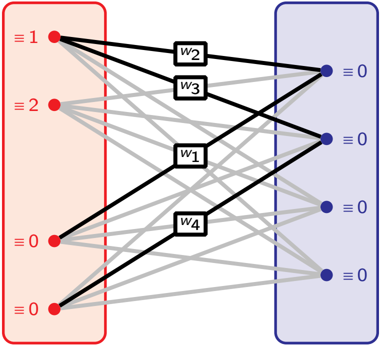

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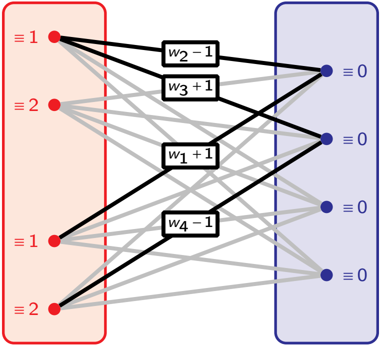

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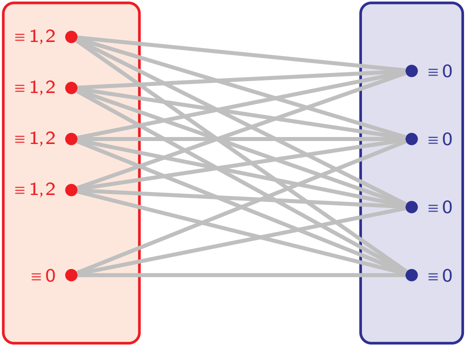
Proof. If $|A|$ and $|B|$ are odd ☹ ... but can reach:



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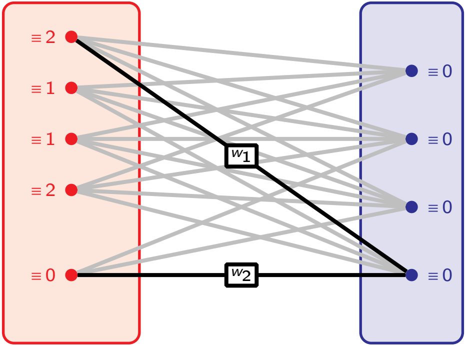

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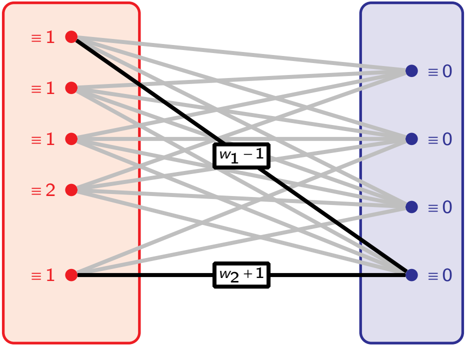

- Proof applies to 3-chromatic graphs with partite sets A, B, C:
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- More generally, k-chromatic graphs, $k \geq 3$ odd, with partite sets $S_0, \ldots, S_{k-1}$:
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- k-chromatic graphs, $k \geq 4$ even, same trick as bipartite graphs

1-2-3 Conjecture
– Other results –

- In general, using $\{1,2,3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi^e_\Sigma \leq 2$ is NP-complete [Dudek, Wajc, 2011]

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- **Q.:** Can we do with using $\{1,\dots,c\}$ for some constant c?
  A.: 30, 16, 13, 6,..., 5! [Kalkowski, Karoński, Pfender, 2012]

1-2-3 Conjecture
– Open questions –

- Prove the 1-2-3 Conjecture for 4-chromatic graphs
  - Done for 4-edge-connected 4-chromatic graphs [Wu, Zhang, Zhu, 2017]

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- Prove that $\chi_\Sigma^e(G) \leq 4$ for every nice graph G
  - Done for 5-regular graphs [B., 2019]
  - Generalized to regular graphs [Przybyło, 2019+]

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- List variants?
  - Every graph is (2,3)-choosable [Wong, Zhu, 2016]
  - No constant bound for the edge version ☹

Going to digraphs

Going to Digraphs

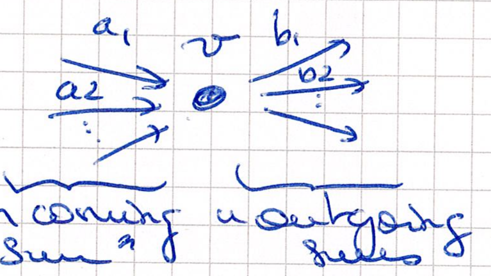
(I)

Wonder how the 123C generalizes to digraphs. Would like some challenge.

In particular:

- same effect when weighting arcs.
- Inductive arguments hard to work out.
- etc...

~~Of course~~ Several options, because "two sums" for the vertices:



Of course, asking $\sigma^- + \sigma^+ \neq \sigma^- + \sigma^+$ is just the 123 Conjecture :)

First work: "Relative sums" $|\sigma^- - \sigma^+| \neq |\sigma^- - \sigma^+|$ — Borowiecki, Grytczuk, Pilśniak 2012

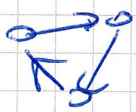
1,2 suffice. Also list version — Khatirinejad, Naserasr, Newman, Seamone, Stevens 2011.

Here, focus on disting. one of $\sigma^-(u), \sigma^+(u)$ and one of $\sigma^-(v), \sigma^+(v)$

1 $(+,+)$ $\overset{u}{\circ} \longrightarrow \overset{v}{\circ}$ $\sigma^+(u) \neq \sigma^+(v)$ — Baudon, B, Sopena 2015

all digraphs can be weighted

Sometimes need 1,2,3



1,2,3 works! induction

choose v st $d^+(v) \geq d^-(v)$

exists since $\Sigma d^- = \Sigma d^+$

key fact: weighting $\overset{v}{o}\longrightarrow$ only affects $\sigma^+(v)$. Can reach sum in $d^+, \dots, 3d^+$, which is $3d^+ - d^+ + 1 = 2d^+ + 1$ values so there is one avoiding that of the neighb ∎ (2)

And deciding the index ≤ 2 is NPC.

$(-,-)$ same up to reversing the arcs.

2017 Woź Przy. Basne B

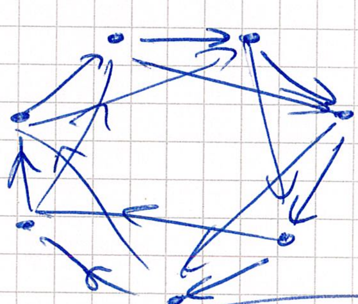
$(+,-)$ $\overset{u}{\bullet}\longrightarrow\overset{v}{\bullet}$ Note: "true behaviour" when weighting an arc

Not all arcs can be weighted: see lonely arcs:

$\rightrightarrows\overset{u}{o}\longrightarrow\overset{v}{o}\rightrightarrows$ but ok if no LA.

$d^+(u) = d^-(v) = 1$

Note: Sometimes '3':



1, 2, 3 actually suffice!

Proof: Bip graph $B(D)$ associated to D: $V^- \cup V^+$

- Explode vertices v to v^-, v^+
- $\forall u \to v$, add edge u^+v^-!

$\Rightarrow$ Equivalence between weighting D and $B(D)$ 'cause $\sigma^-(v) = \sigma(v^+)$ & $\sigma^-(v) = \sigma(v^-)$

Since the 1-2-3 C. holds for bip... ∎

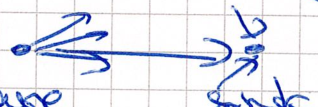
Note: shows that index ≤ 2 iff $B(D)$ is a bip. graph w/ $\chi_\Sigma \leq 2$. Can be checked in polytime due to a characterization (Thomassen, Wu, Zhang).

(−, +) :   Hornák
Przybyło   2018   (3)
Woźniak

Weirdest variant 'cause weighting u→v does not affect the sums of interest…

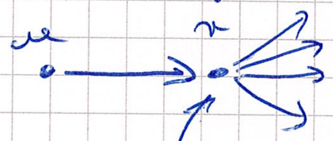
Not defined for all digraphs: ∄

SS arc



$\sigma^-(u) = 0 = \sigma^+$

Also not bounded if LA:

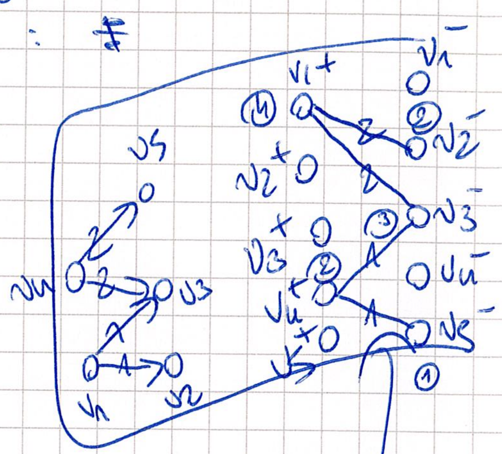


σ⁻ only determined by u→v…

But otherwise, conjecture of 1,2,3. Sometimes need 3.

Theorem: HPW: 4 work.

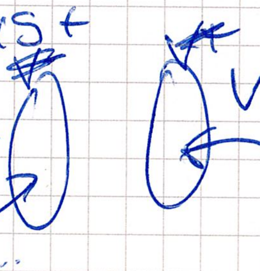
Idea use B(D) again.
Good in terms of sums…
… but does not depict the constraints…



Solution: Regardless of the constraints weight B(D) so that no vertex of V^- has the same sum as in V^+.

1,3 ← 2,4 mod 4…

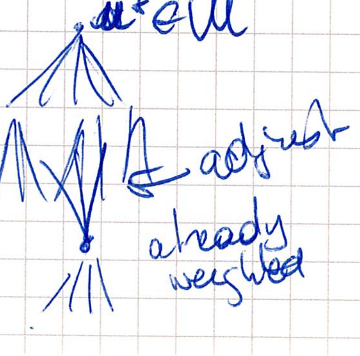
Proof of 1,2,3: w/ Kasper 2015+
same but refined sum…

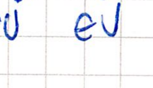


0,3,
4,7,10,13,16…

0,1,2
5,6
8,9
11,12
14,15…

Done by RevBFS… rooted at $u^* \in M$ deg ≥ 2

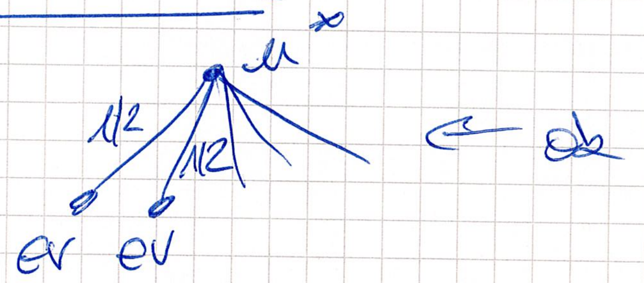


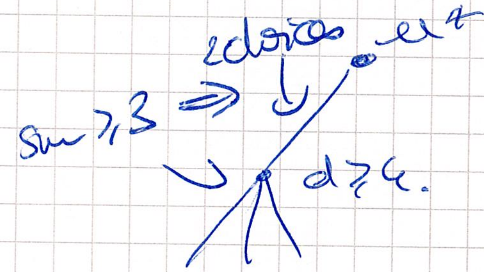


Pbm: u* is "forced"

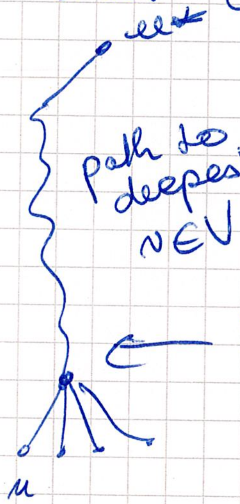
Sol: Show that choices at the last edges.

For instance:





in other cases, no quite choice...



← for that guy, choices. Proof can make these choices go upwards.

And just do that w/ two paths. ☒

Also, E2 is NPC

CCL: Really nice that such different behaviours, different complexity, and proof arguments..
.. but deceiving regarding having a challenging pbm in digraphs...