A Decompositional Approach to the 1-2-3 Conjecture

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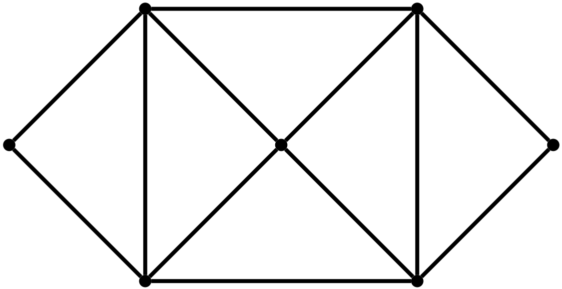
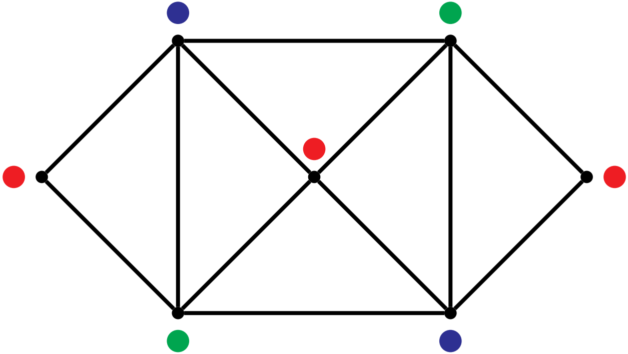
May 4, 2018

General introduction

Make adjacent vertices distinguishable?



Make adjacent vertices distinguishable? ⇒ Proper vertex-colouring ☺



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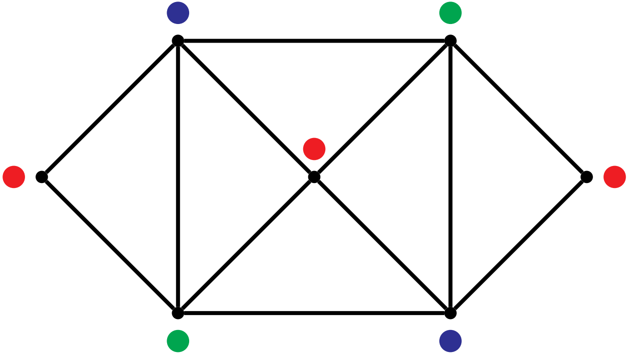

⚠ χ might be as high as $\Delta + 1$ (Brooks' Theorem)

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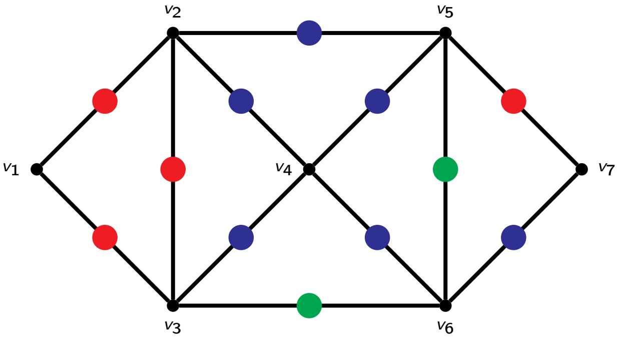

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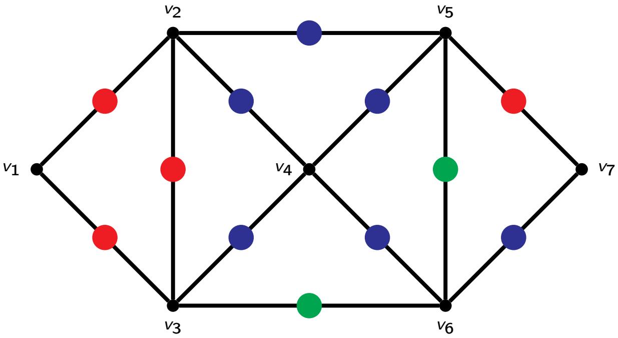

$\mathrm{Col}(v_i)$:= Set of colours "incident" to v_i:

$$\mathrm{Col}(v_1) = \{{\color{red}\bullet}\} \quad \mathrm{Col}(v_2) = \{{\color{red}\bullet}, {\color{blue}\bullet}\} \quad \mathrm{Col}(v_3) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\}$$

$$\mathrm{Col}(v_4) = \{{\color{blue}\bullet}\} \quad \mathrm{Col}(v_5) = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_6) = \{{\color{blue}\bullet}, {\color{green}\bullet}\} \quad \mathrm{Col}(v_7) = \{{\color{red}\bullet}, {\color{blue}\bullet}\}$$

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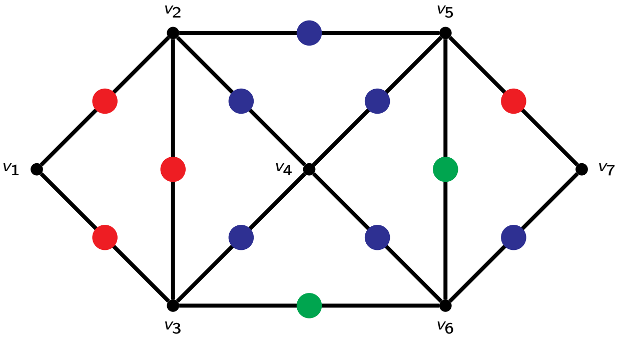

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Neighbours are distinguished!

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- etc.

⇒ Dozens and dozens variants...

A Dynamic Survey of Graph Labeling

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Mathematics Subject Classifications: 05C78

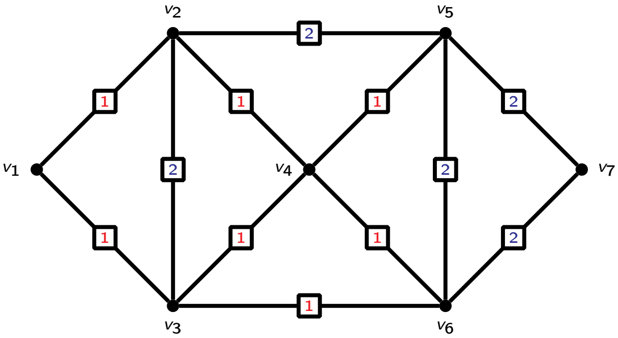
Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the mid 1960s. In the intervening 50 years over 200 graph labelings techniques have been studied in over 2500 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

1-2-3 Conjecture

– Introduction –

Edge-colours = Edge-weights
$\mathrm{Col}(v_i) = \sigma(v_i) :=$ Sums of weights "incident" to v_i



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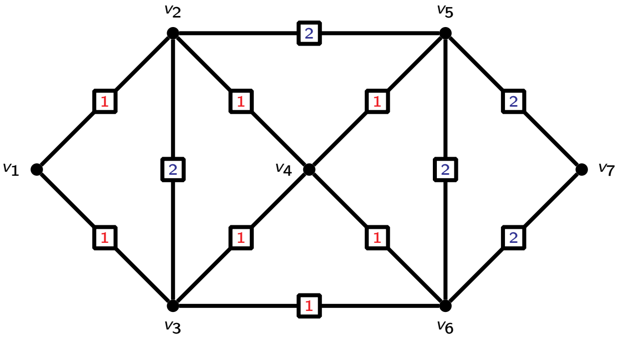

$\sigma(v_1) = 2$ $\sigma(v_2) = 6$ $\sigma(v_3) = 5$ $\sigma(v_4) = 4$
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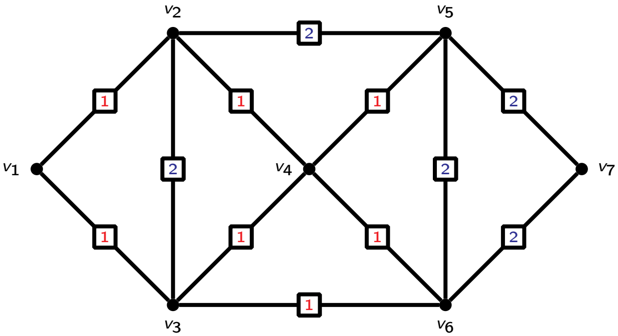

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$\chi^e_\Sigma = 2$ while $\chi = 3$ ☺

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$\chi_{\Sigma}^{e}(G)$ = smallest k such that G has n-s-d k-edge-weightings

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Nice graph = no K_2 as a component

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1-2-3 Conjecture [Karoński, Łuczak, Thomason, 2004]

For every nice graph G, we have $\chi_\Sigma^e(G) \leq 3$.

Edge weights and vertex colours

Michał Karoński and Tomasz Łuczak

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, Poland
E-mail: karonski@amu.edu.pl and tomasz@amu.edu.pl

and

Andrew Thomason

DPMMS, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, England
E-mail: a.g.thomason@dpmms.cam.ac.uk

Received 24th September 2002

Can the edges of any non-trivial graph be assigned weights from $\{1, 2, 3\}$ so that adjacent vertices have different sums of incident edge weights?
We give a positive answer when the graph is 3-colourable, or when a finite number of real weights is allowed.

This problem is also related to irregular multigraphs

How to Define an Irregular Graph

Gary Chartrand
Paul Erdös
Ortrud R. Oellermann

Q.: regular = same degrees, but irregular = ?
(Note: simple graphs with ≥ 2 vertices of unique degrees do not exist)

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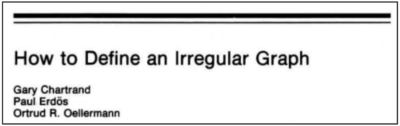
An additional motivation

This problem is also related to irregular multigraphs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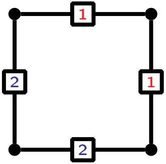
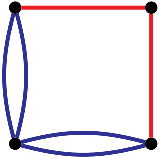
Connection to n-s-d edge-weightings:





$\Rightarrow$ Finding $\chi^{e}_{\Sigma}(G)$ $\Leftrightarrow$ Perform this with minimizing maximum edge multiplication

1-2-3 Conjecture
– Some families of graphs –

Theorem

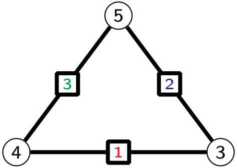
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Make a guess ☺

Theorem

For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. By induction on n. For $n = 3$:



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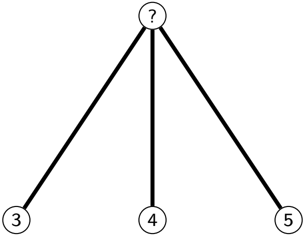
For every $n \geq 3$, we have $\chi^e_\Sigma(K_n) = 3$.

Proof. $n = 4$:



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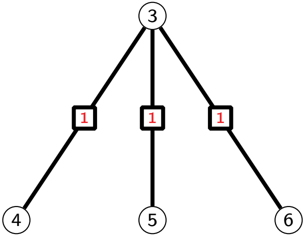

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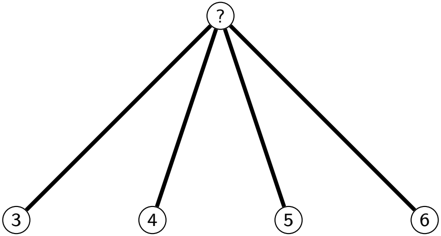
Proof. $n = 5$:



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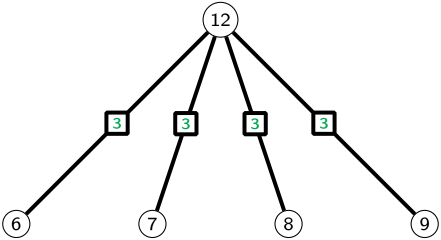

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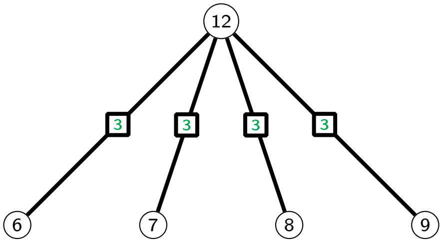

General case: n even $\Rightarrow$ 1's. n odd $\Rightarrow$ 3's.

■

Theorem

For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Any idea ☺ ?

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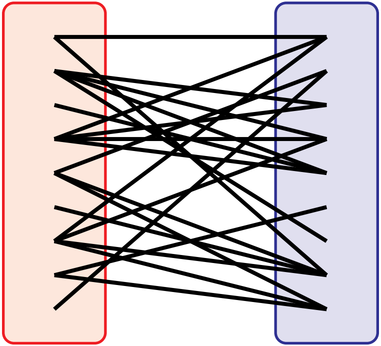
Proof. Bipartition (A, B)



Aim: 3-edge-weighting where $\sigma(A) \equiv 1, 2 \pmod 3$ and $\sigma(B) \equiv 0 \pmod 3$
$\Leftrightarrow$ $\{0, 1, 2\}$-edge-weighting with the same properties

Theorem

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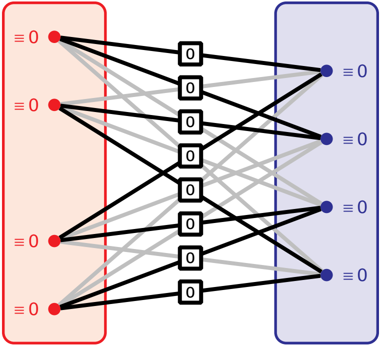
Proof. Assume $|A|$ is even. Start with weights 0. Second condition fulfilled by B.



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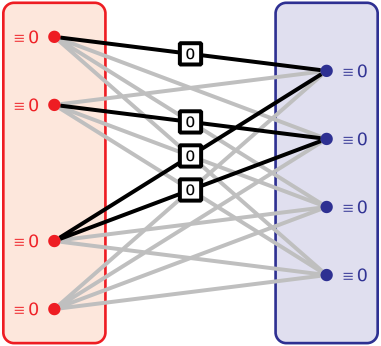
For every nice bipartite graph G, we have $\chi^e_\Sigma(G) \leq 3$.

Proof. Pick a path from A to A with new ends, and apply $+1, -1, \ldots$ along



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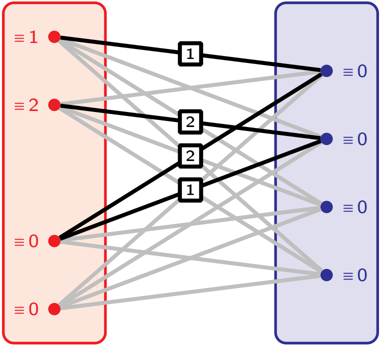

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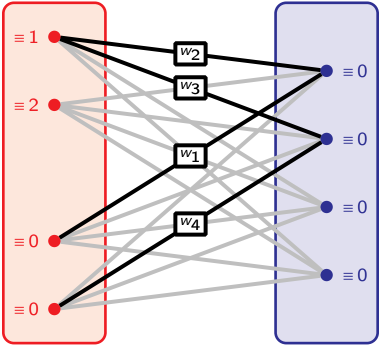
Proof. Repeat until A fulfils the first condition



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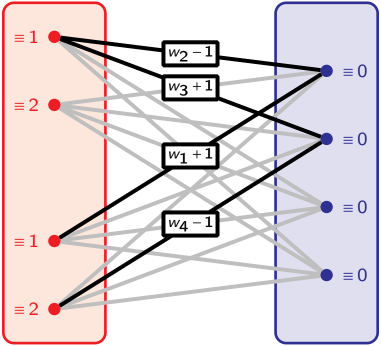

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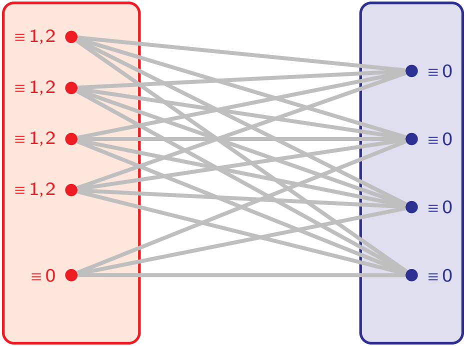
Proof. If $|A|$ and $|B|$ are odd ☹ ... but can reach:



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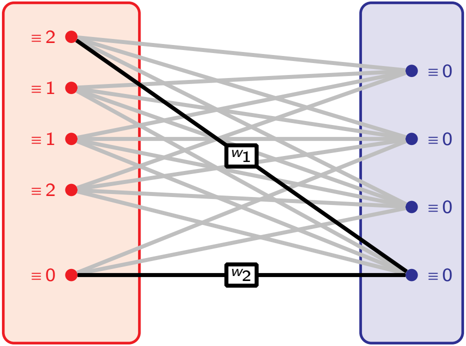
For every nice bipartite graph G, we have $\chi^{e}_{\Sigma}(G) \leq 3$.

Proof. Eventually apply $+1, -1, \ldots$ or conversely towards another vertex in A



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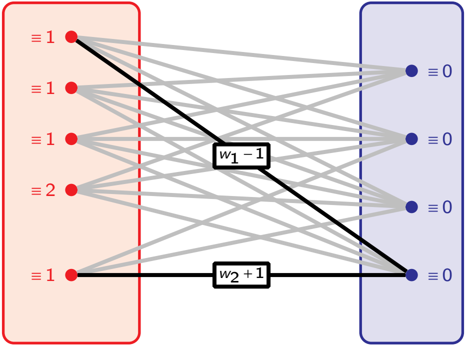

- Proof applies to 3-chromatic graphs with partite sets A, B, C:
  - Use weights $0, 1, 2$
  - Aim $\sigma(A) \equiv 0 \pmod 3$, $\sigma(B) \equiv 1 \pmod 3$, $\sigma(C) \equiv 2 \pmod 3$

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- More generally, k-chromatic graphs, $k \geq 3$ odd, with partite sets $S_0, \ldots, S_{k-1}$:
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  - Aim $\sigma(S_i) \equiv i \pmod k$ for $i = 0, \ldots, k-1$
- k-chromatic graphs, $k \geq 4$ even, same trick as bipartite graphs

- In general, using $\{1,2,3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi^e_\Sigma \leq 2$ is NP-complete [Dudek, Wajc, 2011]

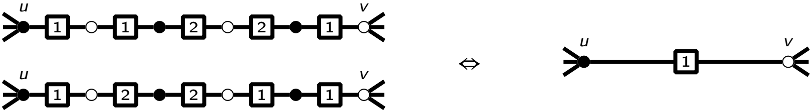
More on the 1-2-3 Conjecture

- In general, using $\{1,2,3\}$ is best possible!
  - Examples: complete graphs, some cycles, etc.
  - Deciding whether $\chi^e_\Sigma \leq 2$ is NP-complete [Dudek, Wajc, 2011]

- **Q.:** Is this true for bipartite graphs?
  A.: $\chi^e_\Sigma(\text{Bipartite}) = 3$: *odd multicacti* [Thomassen, Wu, Zhang, 2016]

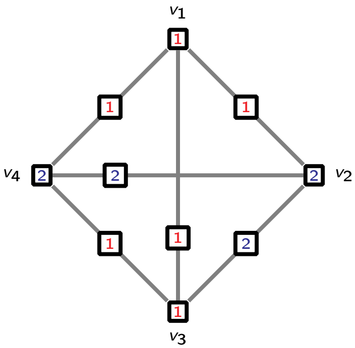
> These graphs can also be described in another way as follows. Take a collection of simple cycles each of length 2 modulo 4 and each with edges colored alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity ≥ 1. The graph with one edge and two vertices is also called an odd multi-cactus.

Intuition: Essentially, with $\{1,2\}$, paths of length $\equiv 1 \pmod 4$ act as edges:

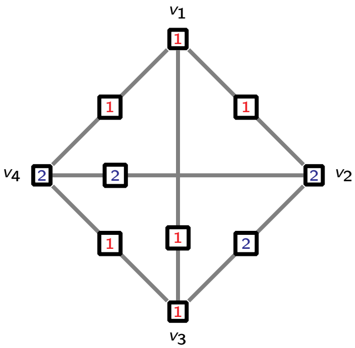


1-2-3 Conjecture
– Best bound –

Best bound on χ^e_Σ obtained from one for a total variant of the problem



Best bound on χ_Σ^e obtained from one for a total variant of the problem



$\sigma(v_1) = 4 \quad \sigma(v_2) = 7 \quad \sigma(v_3) = 5 \quad \sigma(v_4) = 6$

($\sim$ adding a loop at each vertex)
$\chi_\Sigma^t(G)$ = smallest k such that G has n-s-d k-total-weightings

Remarks:

- $\chi^t_\Sigma(G)$ defined for all G
- $\chi^t_\Sigma(G) \leq \chi^e_\Sigma(G)$ for every G

Remarks and a 1-2 Conjecture

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On a 1, 2 Conjecture

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received February 12, 2008, accepted February 3, 2010.

Let us assign positive integers to the edges and vertices of a simple graph G. As a result we obtain a vertex-colouring of G with integers, where a vertex colour is simply a sum of the weight assigned to the vertex itself and the weights of its incident edges. Can we obtain a proper colouring using only weights 1 and 2 for an arbitrary G?

We give a positive answer when G is a 3-colourable, complete or 4-regular graph. We also show that it is enough to use weights from 1 to 11, as well as from 1 to $\lfloor \frac{\chi(G)}{2} \rfloor + 1$, for an arbitrary graph G.

Keywords: neighbour-distinguishing total-weighting, irregularity strength

1-2 Conjecture [Przybyło, Woźniak, 2010]

For every graph G, we have $\chi^t_\Sigma(G) \leq 2$.

Theorem [Kalkowski, 2009]

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    ($\phi(v_i)+1$ = eventual sum, $\phi(v_i)$ the only allowed different sum)
    ⚠ Make sure that $\phi(v_i) \neq \phi(v_j)$ for every backward edge!

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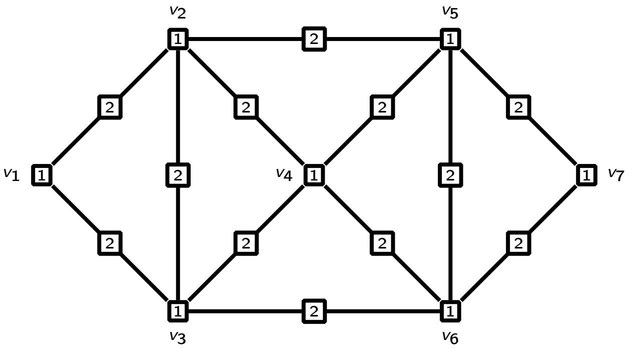
  - Make "valid" weight changes backwards so that $\sigma(v_i) \in \{\phi(v_i), \phi(v_i)+1\}$
- Eventually, do +1 on every vertex weight where $\sigma(v_i) = \phi(v_i)$

Note: Actually, only 1, 2 are used as vertex weights

Vertex ordering: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$



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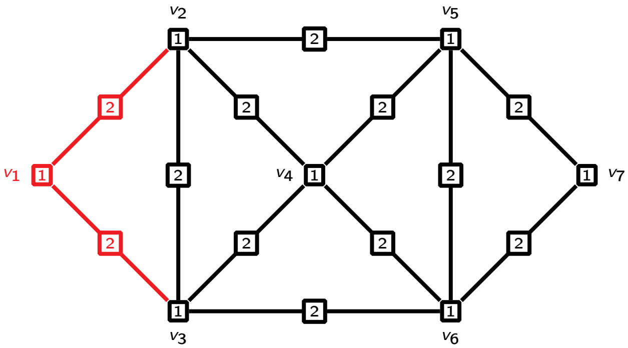

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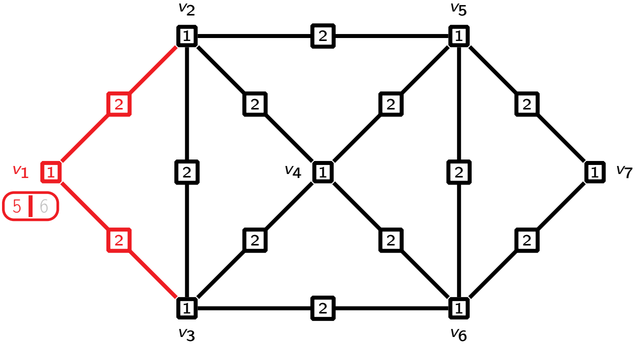

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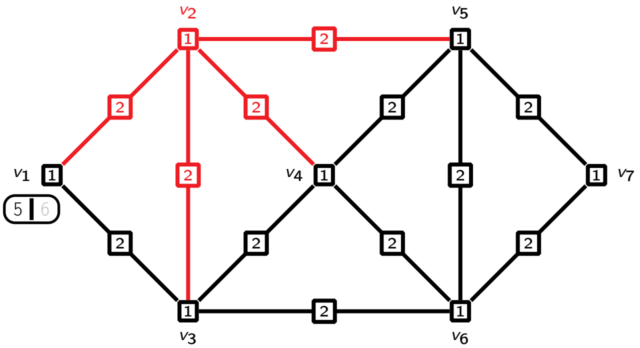

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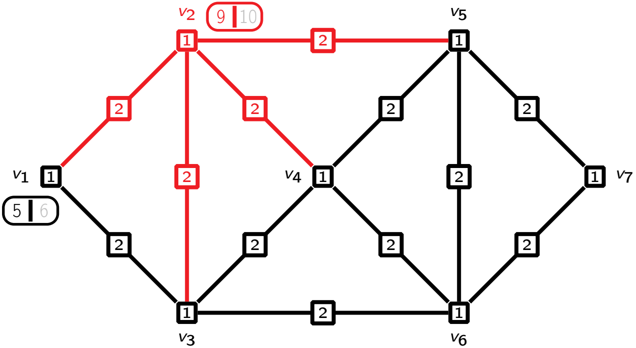

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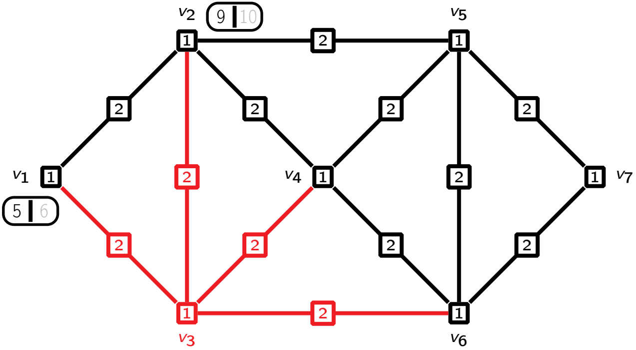
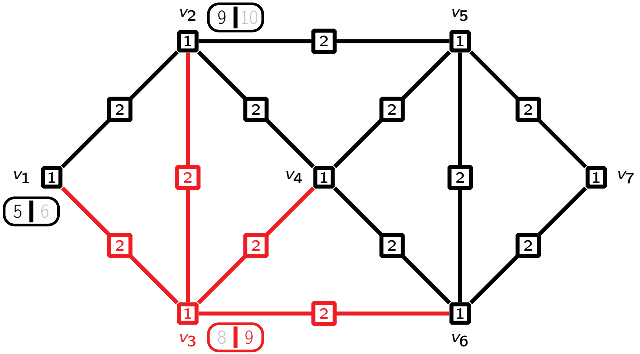


Vertex ordering: $v_1, v_2, \color{red}{v_3}, v_4, v_5, v_6, v_7$



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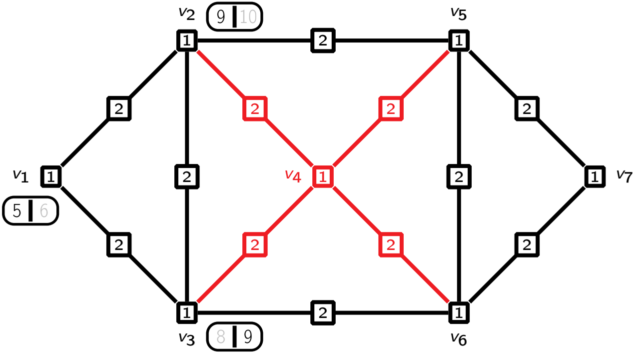

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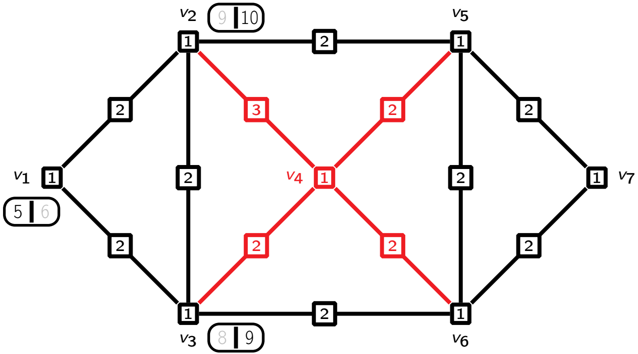

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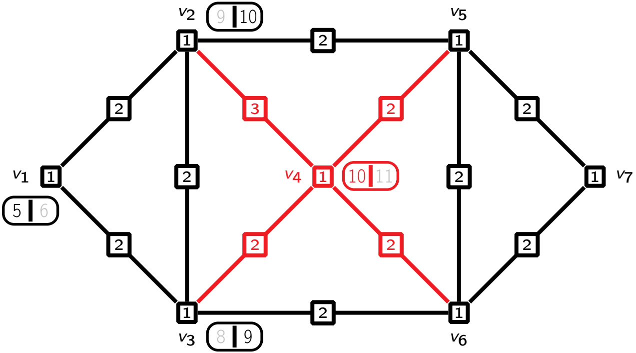

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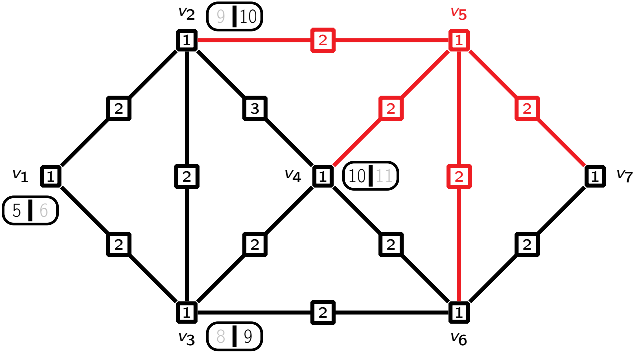

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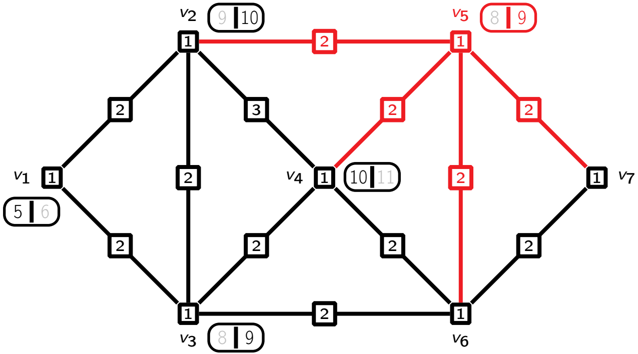

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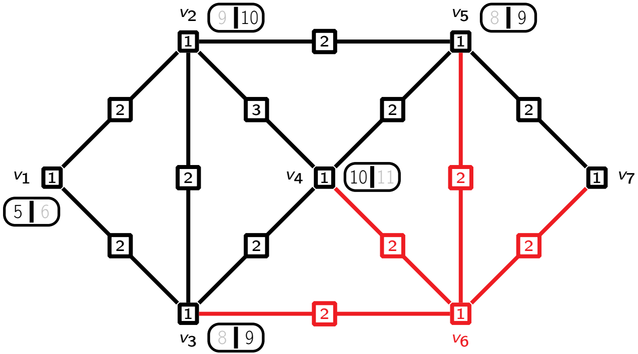

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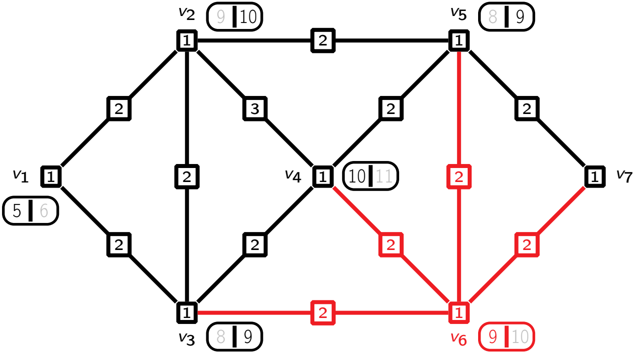

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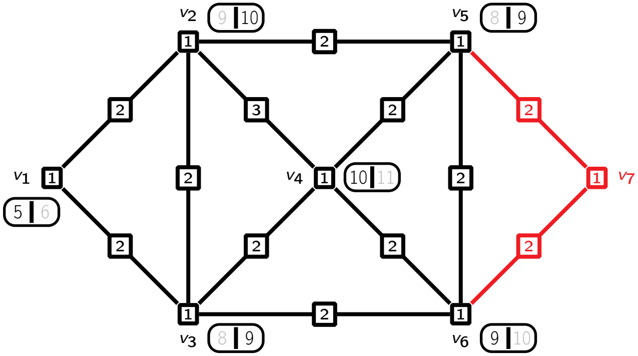

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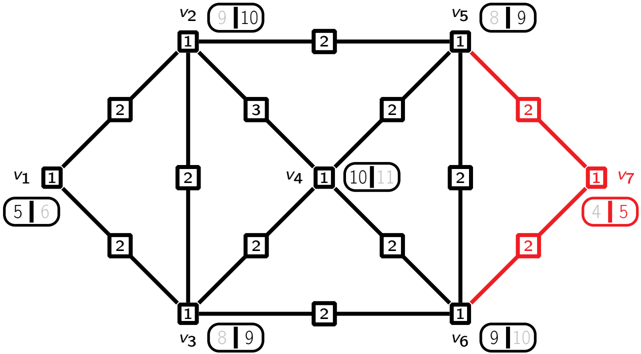
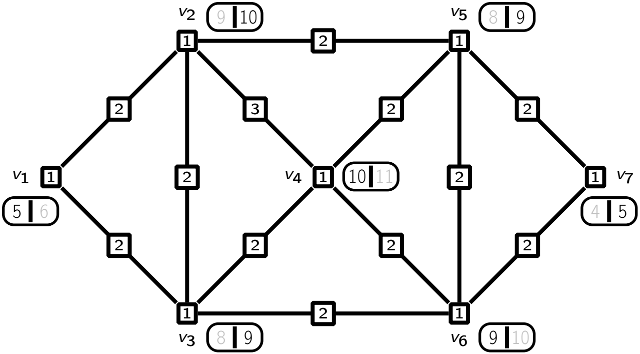
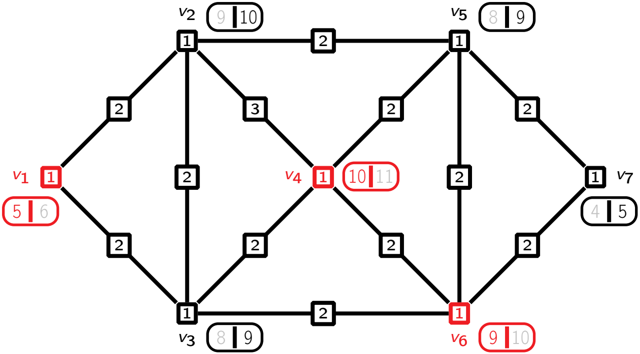
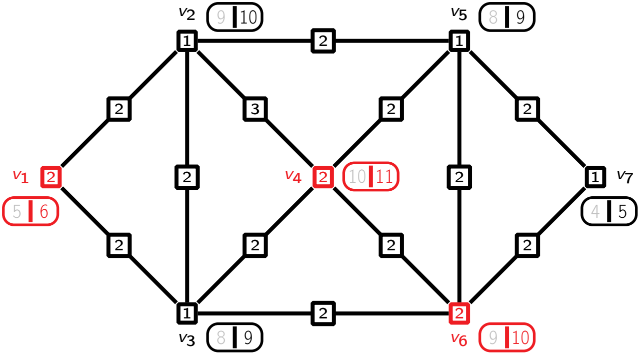


Kalkowski's Algorithm: Final adjustments

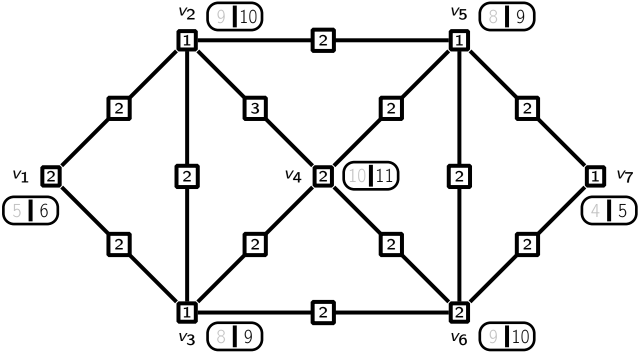




v_2
9 | 10
v_5
8 | 9
1
2
1
2
3
2
2
v_1
1
5 | 6
2
v_4
1
10 | 11
2
1
v_7
4 | 5
2
2
2
2
1
2
1
v_3
8 | 9
v_6
9 | 10



v2
9 | 10
v5
8 | 9
1
2
1
2
3
2
2
v1
2
2
v4
2
10 | 11
2
1
v7
5 | 6
4 | 5
2
2
2
2
1
2
2
v3
8 | 9
v6
9 | 10



v2
9 10
v5
8 9
1
2
1
2
3
2
2
v1
2
5 6
2
v4
2
10 11
2
1
v7
4 5
2
2
2
2
1
2
2
v3
8 9
v6
9 10

- Works because:
  - All edge weight changes are done backwards
  - ⇒ When treating v_i, every backward edge $v_j v_i$ is weighted 2
  - ⇒ A valid change (-1 or +1) per backward edge
  - ⇒ # of candidates as $\phi(v_i) + 1 >$ # of backward neighbours

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  - ⚠ Valid changes backwards are trickier

1-2-3 Conjecture
– Open questions –

- Prove the 1-2-3 Conjecture for 4-chromatic graphs
  - Done for 4-edge-connected 4-chromatic graphs [Wu, Zhang, Zhu, 2017]

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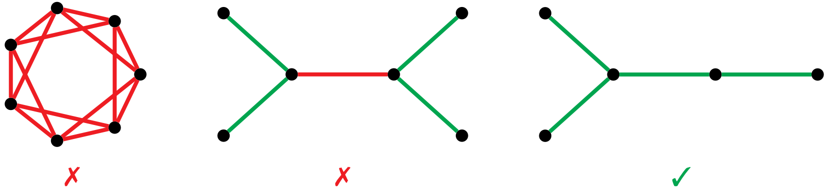
- Prove that $\chi^e_\Sigma(G) \leq 4$ for every nice graph G
- Prove/Disprove the 1-2 Conjecture
- List variants?
  - Every graph is (2,3)-choosable [Wong, Zhu, 2016]
  - No constant bound for the edge version ☹

Locally irregular decompositions
– Introduction –

Locally irregular = Every two adjacent vertices have distinct degrees



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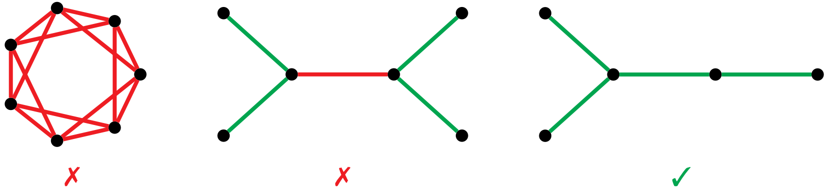

Decomposition of G = Partition $E_1, \ldots, E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, locally irregular edge-colouring)

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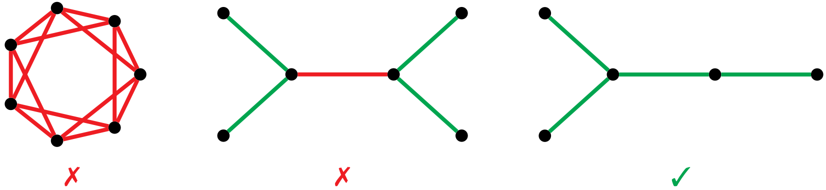
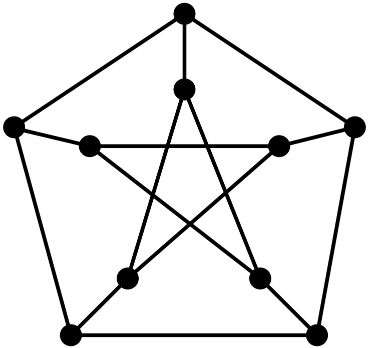
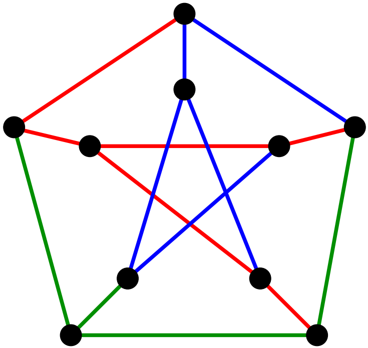


Decomposition of G = Partition $E_1, ..., E_k$ of $E(G)$
Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, locally irregular edge-colouring)

$\chi'_{\rm irr}(G)$ = Smallest $k \geq 1$ s.t. G has locally irregular k-edge-colourings
G decomposable = $\chi'_{\rm irr}(G)$ exists
G exceptional, otherwise



Example



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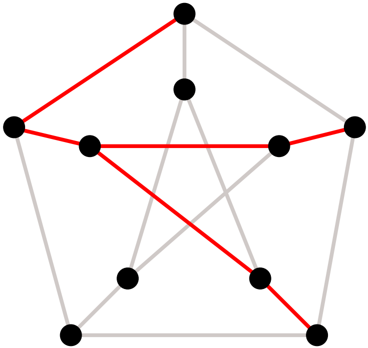

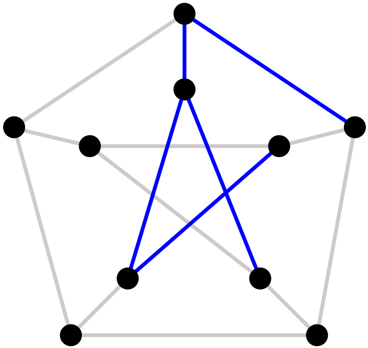

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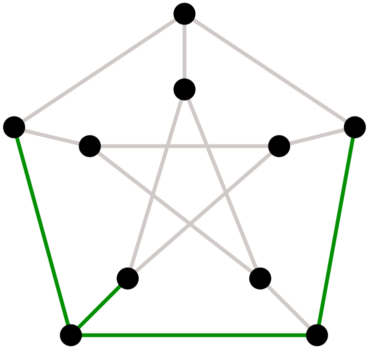

1. Local irregularity = Possible antonym notion to regularity
2. χ'_{irr} = Measure of closeness to irregularity

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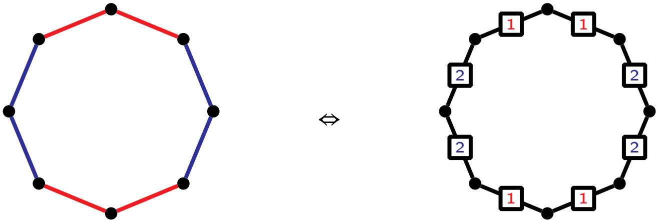
3. Connections and applications to the 1-2-3 Conjecture



In regular graphs, $\chi^e_\Sigma = 2$ **if and only if** $\chi'_{\mathrm{irr}} = 2$

Exceptional graphs?

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Some obvious ones: odd-length paths and odd-length cycles...

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... but also $\mathscr{T}$:

Every connected graph of even size can be decomposed into paths of length 2 and is thus decomposable. Hence, all exceptional graphs have odd size and a complete characterisation of exceptional graphs was given by Baudon, Bensmail, Przybyło, and Woźniak [1]. To state this characterisation, we first need to define a family $\mathcal{T}$ of graphs. The definition is recursive:

- The triangle K_3 belongs to $\mathcal{T}$.
- Every other graph in $\mathcal{T}$ can be constructed by (1) taking an auxiliary graph F being either a path of even length or a path of odd length with a triangle glued to one of its ends, then (2) choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex, say v, of degree 2 in G, and finally (3) identifying v with a vertex of degree 1 of F.

In other words, the graphs in $\mathcal{T}$ are obtained by connecting a collection of triangles in a tree-like fashion, using paths with certain lengths, depending on what elements these paths connect. Let us point out that all graphs in $\mathcal{T}$ have maximum degree 3, have odd size, and all of their cycles are triangles.

Theorem [Baudon, B., Przybyło, Woźniak, 2015]

Exceptional graphs are exactly these three classes of graphs.

How large can χ'_{irr} be?

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Conjecture [Baudon, B., Przybyło, Woźniak, 2015]

For every decomposable graph G, we have $\chi'_{\mathrm{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2}, K_n, etc.). Actually, unless P = NP, no "good" characterization of when $\chi'_{\mathrm{irr}}(G) \leq 2$ [Baudon, B., Sopena, 2015].

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Conjecture verified for:

- trees, regular bipartite graphs, $K_{n,m}$, K_n, some Cartesian products, regular graphs with degree $\geq 10^7$ [Baudon, B., Przybyło, Woźniak, 2015]

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- graphs with $\delta \geq 10^{10}$ [Przybyło, 2016]

Theorem

For every $n \geq 4$, we have $\chi'_{\mathrm{irr}}(K_n) = 3$.

Your turn ☺

Theorem

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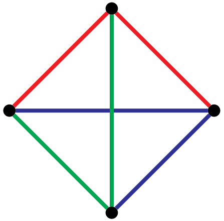
Proof. Quite similar as for the 1-2-3 Conjecture. For $n = 4$:



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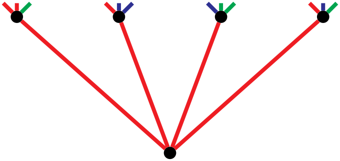
Proof. For $n = 5$:



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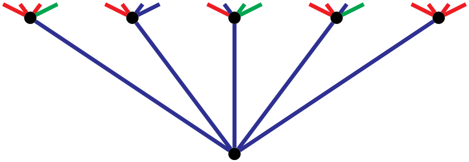

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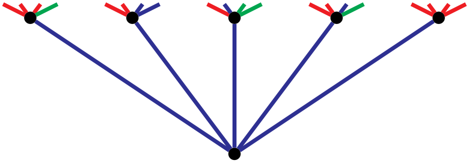
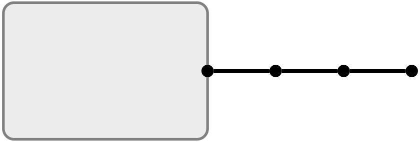


General case: n even $\Rightarrow$ /'s. n odd $\Rightarrow$ /'s. ∎

Theorem

For every decomposable tree T, we have $\chi'_{\text{irr}}(T) \leq 3$.

Proof. For instance, by induction. If a pendant path of length ≥ 3:



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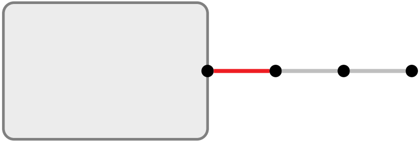

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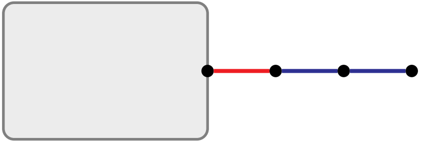

Theorem

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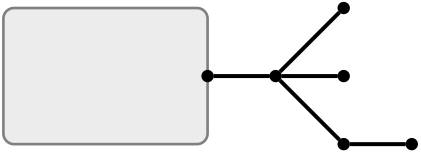
Proof. If a deepest branching node with ≥ 3 children:



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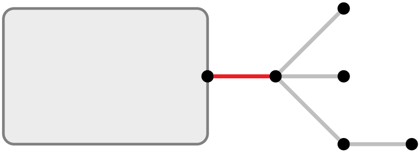

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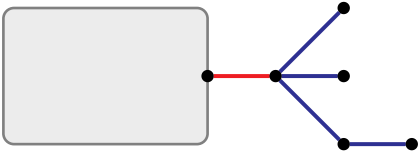

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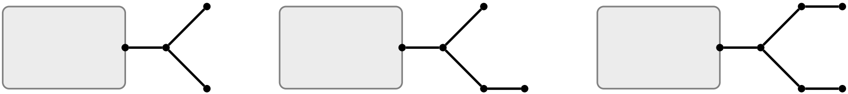

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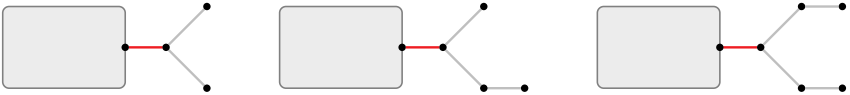

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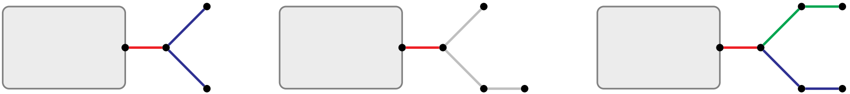

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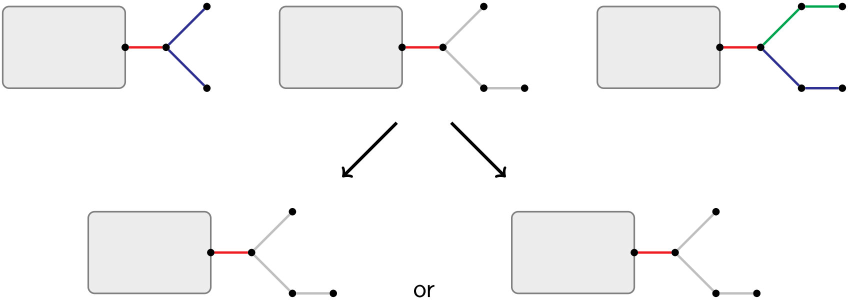

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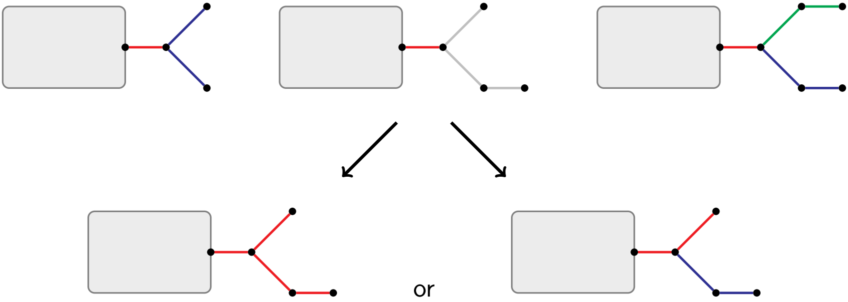

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Theorem [B., Merker, Thomassen, 2016]

For every decomposable bipartite graph G, we have $\chi'_{\text{irr}}(G) \leq 10$.
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General idea: Find edge-disjoint subgraphs $G_1, ..., G_k$ of G s.t.

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   3. $\chi'_{\mathrm{irr}}(G) \leq \chi'_{\mathrm{irr}}(H) + \chi'_{\mathrm{irr}}(D) \leq 3 + 9 \cdot 36 = 327$

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- $\Rightarrow$ $\chi'_{\mathrm{irr}}(\mathrm{decomposable}) \leq 220$!
- (just plug new result in previous approach)

Locally irregular decompositions

– Open questions –

Some open problems/questions

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- Better general bounds via different approaches?

A generalization
– Introduction –

Coloured weights and sums

Each edge → Coloured weight (α,β) w/ colour α and value β
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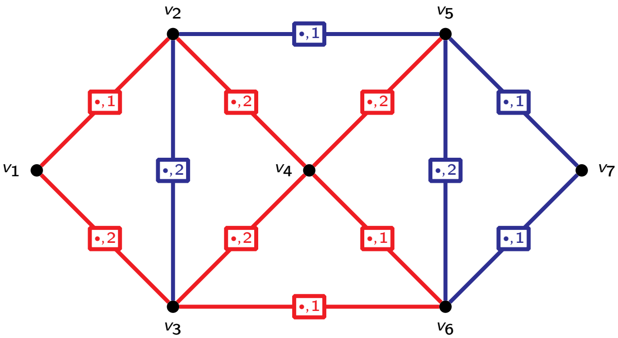

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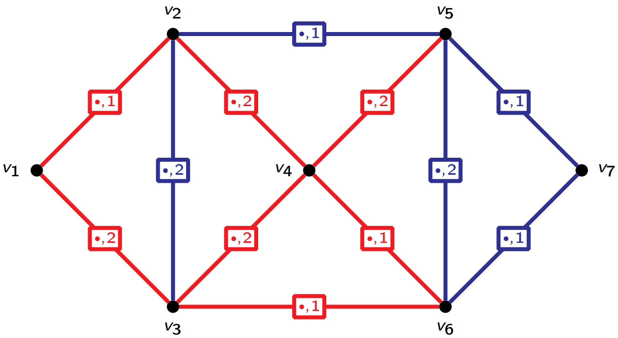

$\sigma_{\bullet}(v_1) = 3$, $\sigma_{\bullet}(v_1) = 0$ $\sigma_{\bullet}(v_2) = 3$, $\sigma_{\bullet}(v_2) = 3$ $\sigma_{\bullet}(v_3) = 5$, $\sigma_{\bullet}(v_3) = 2$

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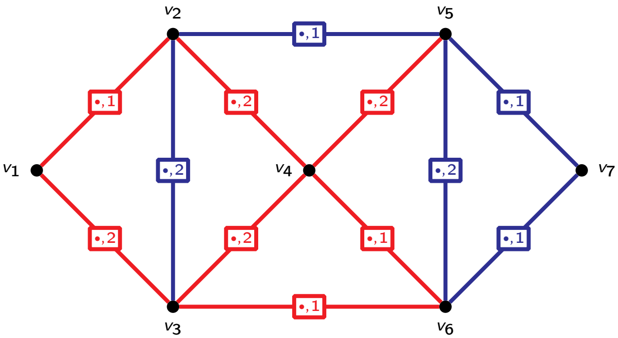

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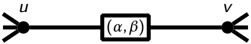
When are adjacent vertices considered distinguished?

Colours $\in \{1, ..., \alpha\}$, Weights $\in \{1, ..., k\}$

Three more or less strong distinction conditions:



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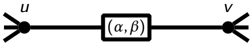

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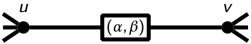

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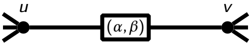

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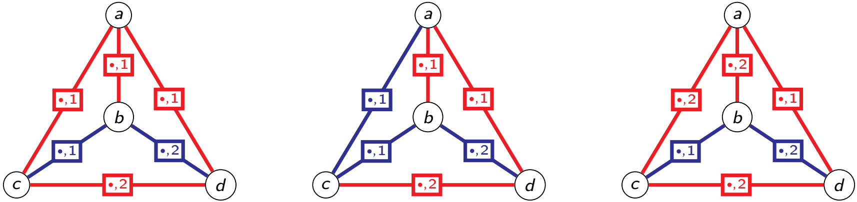
Note: Strong $\Rightarrow$ Standard $\Rightarrow$ Weak; but no converse is true:



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- also, weak $(\ell, 1)$-colouring = ℓ-edge-colouring distinguishing by *multisets*
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  - They are all (4,1)-colourable

Playing with at least two colours and at least two weights?

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nicer graph = no K_2 nor K_3 as connected component

Conjecture [Baudon, B., Davot, Hocquard, Przybyło, Senhaji, Sopena, Woźniak, 2018+]

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Recall: "Strong Conjecture" ⇒ "Standard Conjecture" ⇒ "Weak Conjecture"

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- Bipartite graphs
  ⇒ Proof reduces to odd multicacti

Alternative formulation:

Standard Conjecture

Nicer graphs decompose into two graphs fulfilling the 1-2-3 Conjecture.

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- 9-colourable graphs
  ⇒ Decompositions into two nice 3-colourable graphs

Watch out: When using induction, ⚠ bad components!

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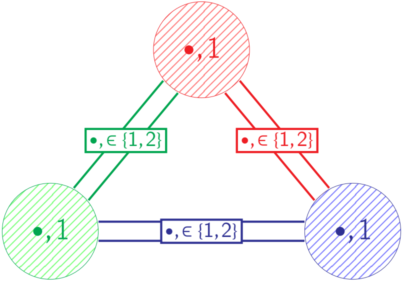

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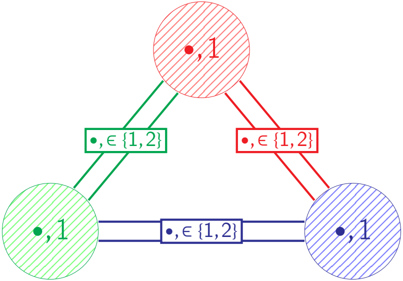

Recall: Results towards the Strong or Standard Conjecture apply

A generalization

– Open questions –

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Thank you for your attention!