The 1-2-3 Conjecture - 10 years after

Ⓘ Motivations & origins

G : simple undirected graph.

G regular → all vertices have the same degree.

"irregularity"? (antonym notion)

G totally irregular → all vertices have distinct degrees

Pb : only K_1 is totally irregular !!

Proof : in a totally irregular simple graph w/ order $n \geq 2$, the degree sequence should be $(0, 1, \ldots, n-1)$. But having an isolated vertex and a universal vertex in a graph is impossible. QED

× Solution 1 [Chartrand, Jacobson, Lehel, Oellermann, Ruiz, Saba - 1988] :

"multiply" the edges of a simple graph to get a totally irregular multigraph.

Example :

Rk : Edge-multiplications ⇒ Adjacencies are preserved.

Pb : Doing this transformation with minimizing the maximum number of times an edge is multiplied?

Nt : For G, this parameter is denoted by $s(G)$, the irregularity strength of G.

Example : $s(C_4) = 3$

Rk1 : $s(K_2) = \infty$

So we consider graphs w/ no isolated K_2's.

Rk2 : Can be seen as a weighting problem: give weights among $\{1, 2, \ldots, k\}$ to edges so that the sums of incident weights at each vertex yield an injective vertex-colouring. So $s(G)$ is the least k s.t. such edge-weightings exist.

theorem [Nierhoff - 2000] :

For every graph G, we have $s(G) \leq |V(G)|+1$

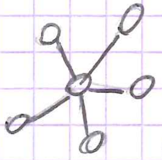
Still some investigations on the irregularity strength, even for trees. Strong dependencies with n_1 and n_2, the number of nodes with degree 1 and 2, resp. But we know e.g. that $s(T)$ is not constantly lower than n_1 (regarding additive p.o.v.)

*** Solution 2 [Chartrand, Erdős, Oellermann - 1988]:**

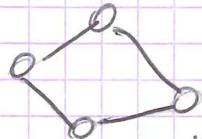
"lower" the notion of irregularity

G locally irregular → all adjacent vertices have distinct degrees.

Examples:



ok!

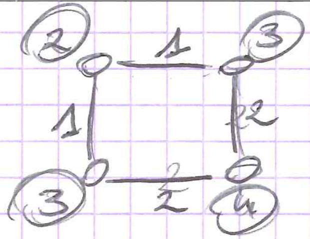


no!

Again, if a simple graph is not locally irregular, then we would like to multiply its edges to get a locally irregular multigraph.

Nt: least number of needed consecutive edge weights / max edge multiplication is denoted $\chi^e_{\Sigma}(G)$ for G.

Example:



$\chi^e_{\Sigma}(C_4) = 2$

Rk: Again $\chi^e_{\Sigma}(K_2) = \infty$.

1-2-3 Conjecture [Karoński, Łuczak, Thomason - 2004]:

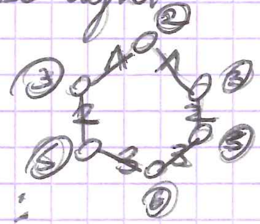
$\chi^e_{\Sigma}(G) \leq 3$ unless G has isolated K_2's.

Said differently, we should be able to "encode" a proper vertex-colouring of every graph using at most 3 edge weights.

II the 1-2-3 Conjecture:

Rk: if true, the "3" would be tight

E.g. $\chi^e_{\Sigma}(C_6) = 3$



But ...

theorem [Dudek, Wajc - 2011]:

Deciding whether $\chi^e_{\Sigma}(G) \leq 2$ is NP-Complete.

However ...

<u>theorem [Addario-Berry, Dalal, Reed - 2008]</u>:

Let G be a random graph from $G_{n,p}$. then a.a.s. $\chi^e_{\Sigma}(G) \leq 2$.

<u>Proof idea</u>: a.a.s. $2\chi(G) < \frac{\delta(G)}{6}$. then pick a subgraph H in which the degrees meet the indexes of a proper vertex colouring (w/ respect to modulo) — Give colour 2 to all edges of H and 1 to the others. <u>QED</u>.

Towards the 1-2-3 Conjecture ...

<u>theorem [Karoński, Łuczak, thomason - 2004]</u>:

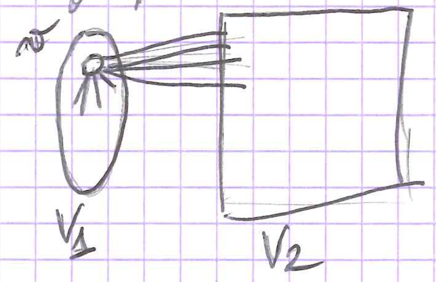
$\chi^e_{\Sigma}(G) \leq 183$.

<u>Proof idea</u>: Partition the vertices with respect to their degrees (small - larger - larger ...). then show by probabilistic arguments that we can edge-weight so that the sums are different. <u>QED</u>

<u>theorem [Addario-Berry, Dalal, McDiarmid, Reed, thomason - 2007]</u>:

$\chi^e_{\Sigma}(G) \leq 30$

<u>Proof idea</u>: degree-constrained spanning subgraph + recursive 8-cuts.



$N_{V_1}(v) \leq N_{V_2}(v)$
⇒ "easy" to distinguish.

Maximum 8-cut — Nice properties:

- for $V_1, V_2, \ldots, V_7$, more "forward" neighb.
- for V_8, lots of neighb. in V_1

then weight inside to reach a particular distinct value modulo "something". Rest is weighted so that the values are different. For V_8, we look at $G[V_1 \cup V_8]$ ⇒ large degree, so degree constrained subgraph. <u>QED</u>.

<u>theorem [Addario-Berry, Dalal, Reed - 2008]</u>:

$\chi^e_{\Sigma}(G) \leq 16$.

<u>Proof idea</u>: Partition into 5 layers w/ more forward neighbours + refined degree-constrained subgraph procedure. <u>QED</u>

Theorem [Wang, Yu - 2008]:
$\chi^e_{\Sigma}(G) \leq 13.$

Proof idea: Basically the same thing. QED.

Finally...
Theorem [Kalkowski, Karoński, Pfender - 2010]:
$\chi^e_{\Sigma}(G) \leq 5.$

Proof: To come =D. QED.

The 1-2-3 Conjecture is verified for many families of graphs such as bipartite graphs, 3-colourable graphs, some planar graphs (discharging method).
Except the conjecture itself, the major open question is:

Question: Is there an "easy" classification of bipartite graphs w/ $\chi^e_{\Sigma} \leq 2$?

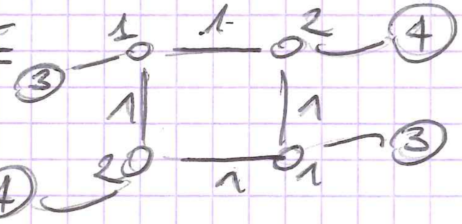
In particular, is the NP-Completeness result true for bipartite graphs?

Ⅲ The 1-2 Conjecture:

Instead of weighting the edges only, also a "local weight" at each vertex (≈ there is a loop at each vertex).

Nt: $\chi^t_{\Sigma}(G)$ = least number of weights needed in a total-colouring of G yielding a proper vertex-colouring (by sums).

Example:



$\chi^t_{\Sigma}(C_4) = 2.$

Rk: χ^t_{Σ} defined for all graphs.

Conjecture [Przybyło, Woźniak - 2010]:
$\chi^t_{\Sigma}(G) \leq 2$ for every graph G.

Rk: $\chi^t_{\Sigma}(G) \leq \chi^e_{\Sigma}(G).$
Rk: If $\chi^t_{\Sigma}(G) \leq k$, then $\chi^e_{\Sigma}(G) \leq 7$.
Proof: (Not complete) Substract 1 on all vertices. Next find a maximum matching between vertices w/ weight 1, and add 1 to

the edge they share. Multiply all edge weights by 2. then pick pairs of vertices w/ weight 1, and apply a "+1/−1" on an odd-length path joining them. <u>QED</u>.

Again the 1-2 Conjecture is true for many graph classes (complete, 3-colourable, 4-regular, some planar, etc.).

Major result =

<u>Theorem [Kalkowski – 2008] =</u>

$\chi^t_{\Sigma}(G) \leq 3 \quad \forall G$.

<u>Proof idea</u> = Process the vertices following an arbitrary ordering. Start w/ all edges weighted 2 and vertices weighted 1. Each time a vertex is considered, we "fix" its "final sum" f and define its "current sum" c as $f = c$. At every step, either $f = c$ or $c = f-1$ for each vertex. When processing v, f is chosen

in such a way that no problem with the previously considered f's. If k backward neighbours, at least $k+1$ possible values. At the end, we change a vertex weight if $c = f-1$. <u>QED</u>.

Many consequences ...

$\left\{ \begin{array}{l} \rightarrow \text{"1-2-3-4-5 theorem"} \\ \rightarrow \text{irregularity strenght} \end{array} \right.$

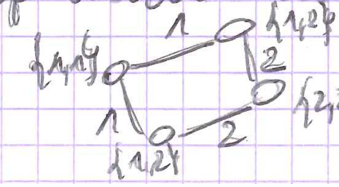
(IV) Derived versions :

We may change the distinguishing aggregate, weight edges and possibly vertices, change the distance between vertices to distinguish...

(A) Multiset versions :

<u>Nt</u> = $\chi^e_m(G)$ least number of edge colours such that we may distinguish the adjacent vertices by their <u>multisets</u> of incident colours.

<u>Example</u> =



$\chi^e_m(C_4) = 2$ ✓

Conjecture [Addario-Berry, Aldred, Dalal, Reed - 2005] :

$\chi^e_m(G) \le 3$ for every graph G.

Rk : Again, would be tight (consider C_6).
And deciding whether $\chi^e_m \le 2$ is NPC.
[Havet, Paramaguru, Sampathkumar - 2012].

Rk : $\chi^e_{\Sigma}(G) \le k \Rightarrow \chi^e_m(G) \le k$.
~~so~~ $\chi^e_m(G) \le 5$ $\forall G$

But ! ...
theorem [Addario-Berry, Aldred, Dalal, Reed - 2005] :

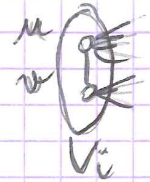
$\chi^e_m(G) \le 4$ $\forall G$.

Proof idea : easy if $\chi(G) \le 3$ (for $\chi^e_{\Sigma} \le 3$)
otherwise a partition

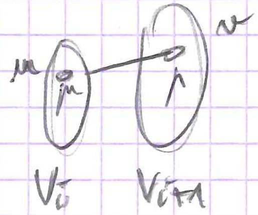
s.t. $\begin{cases} \cdot \forall v \in V_i, \text{ more neighb in } V_{i+1} \\ \cdot \forall v \in V_i, \text{ an edge towards } V_{i+1} \end{cases}$

Colour i to colour all edges of V_i + edges

towards V_{i+1} to distinguish inside V_i. All other edges $\Rightarrow$ extra colour.



+ u & v distinguished by colour i



u & v distingh by colour i+1

QED.

Again the bipartite case is unclear -
But ...
theorem [Havet, Paramaguru, Sampathkumar - 2012] :
G bipartite, $\chi^e_m(G) \le 2$ when $\delta(G) \ge 3$.

Ⓑ Product versions :

Nt : Same as previously but for products of incident weights $\Rightarrow$ χ^e_{π} and χ^t_{π}.

Rk : weight 1 does nothing ...

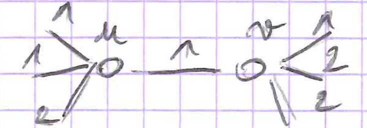
Conjecture [Skowronek - Kaziów - 2012] :

$\chi^e_{\pi}(G) \le 3$ $\forall G$

Ⓒ Locally irregular decompositions

Motivation = we still do not know whether the 1-2-3 Conjecture holds for regular graphs.

Observation = for $\{1,2\}$ only

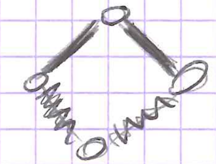


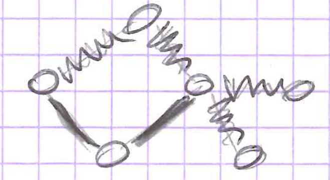
$\Sigma(u) \neq \Sigma(v) \Leftrightarrow \#_a(u) \neq \#_a(v)$

⇒ Subgraphs induced by a- and b-edges must be locally irregular!

Def = locally irregular edge-colouring = each colour class induces a locally irregular subgraph.

Examples =





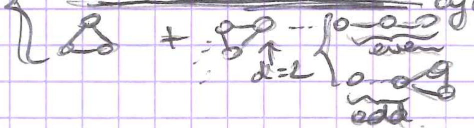
Measure of how far from locally irregular a graph is a graph.

Nt = $\chi'_{irr}(G)$: least # of colours in a locally irregular edge-colouring of G.

Rk = Not only K_2 has $\chi'_{irr} = \infty$!

Theorem [Baudon, B., Przybyło, Woźniak - 2013]

$\chi'_{irr}(G) = \infty \Leftrightarrow$ { G is an odd-length path, cycle,



Proof idea = Induction on $|E(G)|$ + decompo. into P_3's. Successive refinements =

{ × if $\Delta(G) \geq 4$,
× if  independant,
× if "long" cycle.

⇒ $\Delta(G) \leq 3$ +  ~ looks like the family above ⇒ Study of the distance "between" the Δ's

QED

Corollary [Baudon, B., Przybyło, Woźniak - 2013] =

$\chi'_{irr}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor$ unless G is an exception.

Bad because experiments suggest =

Conjecture [Baudon, B., Przybyło, Woźniak - 2013] =

$\chi'_{irr}(G) \leq 3$ unless G is an exception.

Rk = Again $\chi'_{irr}(C_6) = 3$ 

and NPC result [Baudon, B., Sopena - 2015]

Rk = $\chi^e_{mn}(G) \leq \chi'_{irr}(G)$ $\forall G$

Conjecture verified for

× trees ⇒ ≤ 3 reached for ∞ly many trees but good characterization [Baudon, B., Sopena - 2015].

× complete ⇒ ≤ 3
× some bipartite = complete, regular $\delta \geq 3$ ...
× Cartesian product w/ $\chi'_{irr} \leq 3$.

[Baudon, B., Przybyło, Woźniak - 2013]

+ regular graphs w/ degree $\geq 10^7$
(good because "least locally irregular graphs")

theorem [Baudon, B., Przybyło, Woźniak - 2013]:

G regular w/ $\Delta(G) \geq 10^7$, $\chi'_{irr}(G) \leq 3$.

Proof idea = Degree constrained subgraphs + Lovász Local Lemma to get "good" "aimed" degrees. QED.

Question = Again, not sure about bipartite graphs, same questions as prvsly.

theorem [B., Stevens - 2014] =

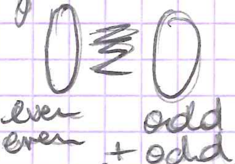
If we "allow" K_2's in decompositions, we get
- $\chi_{irr}(\text{bipartite}) \leq 6$.
- $\chi_{irr}(\text{all}) \leq 6 \log_2(\Delta(G))$

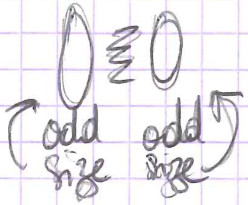
Proof idea =

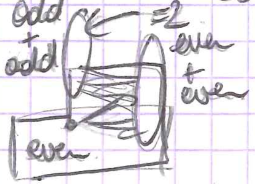
× 1st part = Decompo into Eulerian + Forest
↳ ≤ 4 ↳ ≤ 2

• if 

then decompo into 2 locally irregular



• if 

then 

↳ remove a path to get sth locally irregular
≤ 2
↳ $=1$

× 2nd part = Decompo into $\chi(G)$ bipartite graphs + independent 6-decompo.

QED.

Conjecture [B., Stevens - 2014] =

2 colours should always suffice.

(V) For other kinds of graphs.

(A) the 1-2-3 Conjecture for hypergraphs.

theorem [Kalkowski, Karoński, Pfender - 2013]:

edge size $r \geq 4 \rightarrow \{1, 2, \ldots, r+1\}$ best $r \leq 3 \rightarrow \{1 \ldots 5\}$ suffice
$r \geq 4 \rightarrow \{1 \ldots 5r-5\}$

Proofs use the algorithm for {1,2...5} + vertex deletion to get 2-edges.

B the 1-2-3 Conjecture for directed graphs

1 Potential version

Def : $\chi^e_{\Sigma^+ - \Sigma^-}(\vec{G})$ = least # of weights so that adjacent vertices of $\vec{G}$ get distinguished by "outsum − insum".

theorem [Borowiecki, Grytczuk, Pilśniak - 2012] + [Khatirinejad, Naserasr, Newman, Seamone, Stevens - 2011] :

$$\chi^e_{\Sigma^+ - \Sigma^-}(\vec{G}) \le 2.$$

Proof idea : 1st : chip configuration + exchage

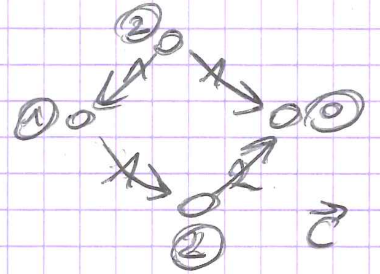
2nd : Combinatorial Nullstellensatz

QED.

2 Outsum version

Def: $\chi^e_{\Sigma^+}(\vec{G})$ = least # of weights so that adjacent vertices of $\vec{G}$ get distinguished by their "outsums".

Example :

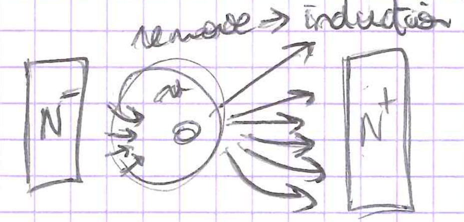


$\chi^e_{\Sigma^+ - \Sigma^-}(\vec{C}) = 2$

theorem [Baudon, B., Sopena - 2015] :

$$\chi^e_{\Sigma^+}(\vec{G}) \le 3 \quad \forall \vec{G}$$

Proof : Choose v with $d^+(v) \ge d^-(v)$



$2d^+(v)+1$ outsums for v, but only $\le 2d^+(v)$ neighbours. Weighting does not alter the other outsums. QED.

Rk : Argument correct for list & product

Graphs w/ $\chi^e_{\Sigma^+} \le 2$?

theorem [Baudon, B., Sopena - 2015] :

It is NPC to decide this.

Rk : the 1-2 Conjecture analogue is wrong :

2-outregular K_5.

③ Locally irregular decomposition

Def = locally irregular oriented graph = adjacent vertices have distinct outdegrees.

Nt = $\chi'_{irr}(\vec{G})$ analogously ...

Conjecture [B., Renault - 2014] =

$$\chi'_{irr}(\vec{G}) \leq 3 \quad \forall \vec{G}$$

Rk = $\chi'_{irr}($  $) = 3$.

theorem [B., Renault - 2014] =

$$\chi'_{irr}(\vec{G}) \leq 6 \quad \forall \vec{G}$$

Proof idea = ~~First~~ show $\chi'_{irr}(\text{acyclic}) \leq 3$

then $\vec{G} = 2 \times \text{acyclic}$

$$\chi'_{irr}(\vec{G}) \leq 2 \times \chi'_{irr}(\text{acyclic}) = 6$$

QED

Rk = 3 may be needed for acyclic.

theorem [B., Renault - 2014] =

Deciding whether $\chi'_{irr}(\vec{G}) \leq 2$ is NPC, even for acyclic graphs.

Ⓥ⃝ⅤI And also =

list of possible weights at each vertex/edge.

× **Conjecture [Bartnicki, Grytczuk, Niwczyk - 200**

the list-1-2-3 Conjecture should be true ...

verified for complete, complete bipartite, trees using the Combinatorial Nullstellensatz.

theorem [Seamone - 2012] :

$$\mathrm{ch}^{e}_{\Sigma}(G) \leq 2\Delta(G) + 1 \quad \forall G$$

× **Conjecture [Personal Com.]** :

the list 1-2 Conjecture should be true.

theorem [Personal Com.] :

$$\mathrm{ch}^{t}_{\Sigma}(G) \leq 3$$